Supplement. The Radón-Riesz Theorem

Note. In Royden and Fitzpatrick's *Real Analysis*, 4th Edition (Boston: Pearson Education, 2010) a proof of the Radón-Riesz Theorem is only given for $p = 2$ (in which case $q = 2$ as well). Here, a proof is given for general p, $1 < p < \infty$. The argument given here is based on Frigyes Riesz and Béla Szőkefnalvy-Nagy's *Functional Analysis* (Mineola, NY: Dover Publishing, 1990; pages 78–80). This is an unabridged republication of the 1955 version of the book.

Note. The Radón-Reisz Theorem states that weak convergence in $L^p(E)$ implies strong convergence in $L^p(E)$ for $1 < p < \infty$ when $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. Recall that "$\{f_n\}$ converges weakly to f" in $L^p(E)$ is denoted $\{f_n\} \rightharpoonup f$.

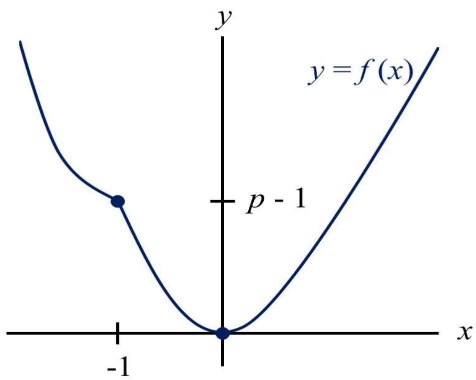
The Radón-Riesz Theorem.
Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$. Then $\{f_n\} \to f$ in $L^p(E)$ if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

Lemma R-R-1. For $p \geq 2$ and for every $x \in \mathbb{R}$, $|1 + x|^p \geq 1 + px + x|x|^p$ for some positive $c < 1$ (independent of x, but possibly dependent on p).

Proof. Consider $f(x) = |a + x|^p - 1 - px$. Then

$$f'(x) = \begin{cases} p|1 + x|^{p-1} - p & \text{for } x > -1 \\ -p|1 + x|^{p-1} - p & \text{for } x < -1. \end{cases}$$

So $f'(x) > 0$ for $x > 0$ and $f'(0) = 0$. Since $f(0) = 0$, then $f(x) > 0$ for $x > 0$. Also, $f'(x) < 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $x < -1$. Since $f(-1) = p - 1 \geq 1$, then the graph of f is similar to (one can show that $f''(x) > 0$ for $x \neq 1$):



So $f(x) > 0$ for all $x \neq 0$. Next, we consider $g_p(x) = \dfrac{|1+x|^p - 1 - px}{|x|^p}$. For $p = 2$, $g_2(x) = \dfrac{(1+x)^2 - 1 - 2x}{x^2} = 1$ if $x \neq 0$. For $p > 2$,

$$\lim_{x \to 0^+} g_p(x) = \lim_{x \to 0^+} \frac{|1+x|^p - 1 - px}{|x|^p} = \lim_{x \to 0^+} \frac{(1+x)^p - 1 - px}{x^p} \overset{0/0}{=} \lim_{x \to 0^+} \frac{p(1+x)^{p-1} - p}{px^{p-1}}$$

$$\overset{0/0}{=} \lim_{x \to 0^+} \frac{p(p-1)(1+x)^{p-2}}{p(p-1)x^{p-2}} = \lim_{x \to 0^+} \frac{(1+x)^{p-2}}{x^{p-2}} = +\infty,$$

and

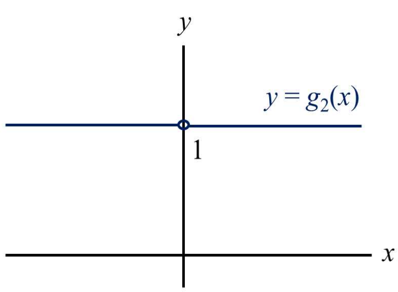
$$\lim_{x \to 0^-} g_p(x) = \lim_{x \to 0^-} \frac{|1+x|^p - 1 - px}{|x|^p} = \lim_{x \to 0^-} \frac{(1+x)^p - 1 - px}{(-x)^p} \overset{0/0}{=} \lim_{x \to 0^-} \frac{p(1+x)^{p-1} - p}{p(-x)^{p-1}(-1)}$$

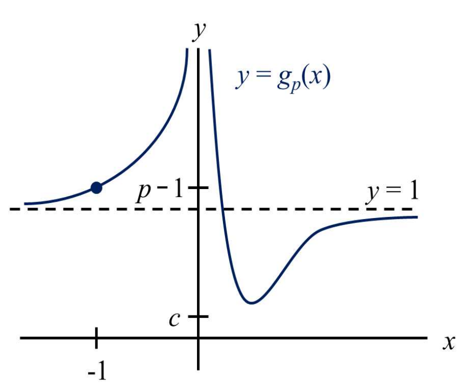
$$\overset{0/0}{=} \lim_{x \to 0^-} \frac{p(p-1)(1+x)^{p-2}}{p(p-1)(-x)^{p-2}} = \lim_{x \to 0^-} \frac{(1+x)^{p-2}}{(-x)^{p-2}} = +\infty.$$

So $\lim_{x \to 0} g_p(x) = +\infty$ for $p > 2$. As $|x| \to \infty$ we have

$$\lim_{x \to \infty} g_p(x) = \lim_{x \to \infty} \frac{|1+x|^p - 1 - px}{|x|^p} = 1.$$

So the graph of g_p is similar to the following:





So there is some c, $0 < c \le 1$, such that $g_p(x) > c$ for all $x \neq 0$. Therefore, $g_p(x) = \dfrac{|1+x|^p - 1 - px}{|x|^p} \ge c$ for all $x \neq 0$, and $|1+x|^p - 1 - px \ge c|x|^p$ for all $x \in \mathbb{R}$ since the result easily holds for $x = 0$. ∎

Note. The graph of $g_2(x)$ shows that we can take $c = 1$ for $p = 2$, and that is why the proof for $p = 2$ is relatively easy (the case $p = 2$ is given by Royden and Fitzpatrick).

Lemma R-R-2. For $1 < p < 2$ and every $x \in \mathbb{R}$, the function

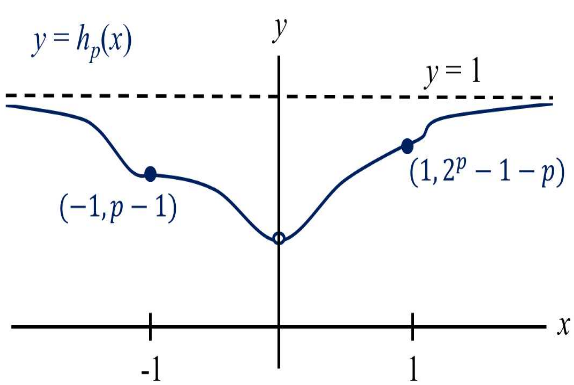
$$h_p(x) = \begin{cases} \frac{|1+x|^p - 1 - px}{|x|^p} & \text{for } |x| > 1 \\ \frac{|1+x|^p - 1 - px}{x^2} & \text{for } |x| \le 1 \end{cases}$$

satisfies $h_p(x) \ge c$ (for $x \neq 0$) for some positive $c < 1$ (independent of x, but possibly dependent on p).

Proof. As shown in the proof of Lemma R-R-1, the numerator of $h_p(x)$ is positive for $x \neq 0$. Now

$$\lim_{h\to 0} h_p(x) = \lim_{x\to 0} \frac{(1+x)^p - 1 - px}{x^2} \overset{0/0}{=} \lim_{x\to\infty} \frac{p(1+x)^{p-1} - p}{2x} \overset{0/0}{=} \lim_{x\to 0} \frac{p(p-1)(1+x)^{p-2}}{2} = \frac{p(p-1)}{2}.$$

Next, h_p is continuous at $x = -1$ and $h_p(-1) = p-1$, h_p is continuous at $x = 1$ and $h_p(1) = 2^p - 1 - p$. As established in Lemma R-R-1, $\lim_{|x|\to\infty} h_p(x) = 1$. Since h_p is continuous on $\mathbb{R}$, then there is a positive c, $c < 1$, with $c \le h_p(x)$ for all $x \in \mathbb{R}$. ∎



Proof of the Radón-Riesz Theorem for General p.

First, suppose $\{f_n\} \to f$ in $L^p(E)$. Notice that, by the Triangle Inequality, $g = g - h + h$ implies that $\|g\|_p \le \|g - h\|_p + \|h\|_p$ and so $\|g\|_p - \|h\|_p \le \|g - h\|_p$. Similarly, $h = h - g + g$ implies $\|h\|_p \le \|h - g\|_p + \|g\|_p$ and so $\|h\|_p - \|g\|_p \le \|h - g\|_p = \|g - h\|_p$. So $|\|g\|_p - \|h\|_p| \le \|h - g\|_p$ for all $h, g \in L^p(E)$. So with $\{f_n\} \to f$ in $L^p(E)$, we have for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$,

$$|\|f_n\|_p - \|f\|_p| \le \|f_n - f\|_p < \epsilon.$$

Therefore, $\lim_{n\to\infty} \|f_n\| = \|f\|$.

Second, suppose $\{f_n\} \rightharpoonup f$ in $L^p(E)$ and $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. We first consider the case $p \ge 2$. By Lemma R-R-1, for all $y \in \mathbb{R}$, $|1 + y|^p \ge 1 + py + c|y|^p$ for some c, $0 < c < 1$. Define $E_0 = \{x \in E \mid f(x) = 0\}$. For $x \in E \setminus E_0$, replace y with $\dfrac{f_n(x) - f(x)}{f(x)}$ to get

$$\left|1 + \frac{f_n(x) - f(x)}{f(x)}\right|^p \ge 1 + p\left(\frac{f_n(x) - f(x)}{f(x)}\right) + c\left|\frac{f_n(x) - f(x)}{f(x)}\right|^p$$

or

$$\left|\frac{f_n(x)}{f(x)}\right|^p \ge 1 + p\left(\frac{f_n(x) - f(x)}{f(x)}\right) + c\left|\frac{f_n(x) - f(x)}{f(x)}\right|^p$$

for $x \in E \setminus E_0$, or

$$|f_n(x)|^p \ge |f(x)|^p + p\frac{|f(x)|^p}{f(x)}(f_n - f(x)) + c|f_n(x) - f(x)|^p$$

$$= |f(x)|^p + p|f(x)|^{p-2} f(x)(f_n(x) - f(x)) + c|f_n(x) - f(x)|^p$$

for $x \in E \setminus E_0$. Notice that this also holds when $f(x) = 0$, since $c < 1$. Therefore this inequality holds for all $x \in E$. So by monotonicity of integration,

$$\int_E |f_n|^p \ge \int_E |f|^p + p\int_E |f|^{p-2} f(f_n - f) + c\int_E |f_n - f|^p. \qquad (*)$$

Now $|f|^{p-2}f = \operatorname{sgn}(f)|f|^{p-1}$ and by Hölder's Inequality ("Moreover..."), $|f|^{p-2}f \in L^q(E)$. Since $\{f_n\} \to f$ in measure, then $\int_E |f|^{p-1} f(f_n - f) \to 0$. By hypothesis, $\int_E |f_n|^p \to \int_E |f|^p$. For for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \ge N$, we have from $(*)$

$$c\epsilon > \int_E |f_n|^p - \int_E |f|^p + p\int_E |f|^{p-2} f(f - f_n) \ge c\int_E |f_n - f|^p.$$

So $\|f_n - f\|_p \to 0$ as $n \to \infty$ and $\{f_n\} \to f$ with respect to the L^p norm for $p \ge 2$.

Third, suppose $1 \le p < 2$. By Lemma R-R-2,

$$\begin{cases} |1+y|^p - 1 - py \ge c|y|^p \text{ for } |y| > 1 \\ |1+y|^p - 1 - py \ge cy^2 \text{ for } |y| \le 1 \end{cases}$$

for some $0 < c < 1$. Define $E_0 = \{x \in E \mid f(x) = 0\}$. Define $E_n = \{x \in E \mid |f_n(x) - f(x)| > |f(x)|\}$. For $x \in E \setminus E_0$, replace y with $\dfrac{f_n(x) - f(x)}{f(x)}$. We have

$$\begin{cases} \left|1 + \frac{f_n(x)-f(x))}{f(x)}\right| - 1 - p\frac{f_n(x)-f(x)}{f(x)} \ge c\left|\frac{f_n(x)-f(x)}{f(x)}\right|^p \text{ for } x \in E_n \setminus E_0 \\ \left|1 + \frac{f_n(x)-f(x))}{f(x)}\right| - 1 - p\frac{f_n(x)-f(x)}{f(x)} \ge c\left(\frac{f_n(x)-f(x)}{f(x)}\right)^2 \text{ for } x \in E \setminus E_n \setminus E_0 \end{cases}$$

or

$$\begin{cases} |f_n(x)|^p \ge |f(x)|^p + p|f(x)|^{p-1}f(x)(f_n(x) - f(x)) + c|f_n(x) - f(x)|^p \text{ for } x \in E_n \setminus E_0 \\ |f_n(x)|^p \ge |f(x)|^p + p|f(x)|^{p-1}f(x)(f_n(x) - f(x)) + c(f_n(x) - f(x))^2|f(x)|^{p-2} \text{ for } x \in E \setminus E_n \setminus E_0 \end{cases}$$

or

$$\begin{cases} |f_n(x)|^p - |f(x)|^p + p|f(x)|^{p-2}f(x)(f(x) - f_n(x)) \ge c|f_n(x) - f(x)|^p \text{ for } x \in E_n \setminus E_0 \\ |f_n(x)|^p - |f(x)|^p + p|f(x)|^{p-2}f(x)(f(x) - f_n(x)) \ge c(f_n(x) - f(x))^2|f(x)|^{p-2} \text{ for } x \in E \setminus E_n \setminus E_0. \end{cases}$$

Notice that these also hold when $f(x) = 0$, since $c < 1$. Therefore the inequalities hold for all $x \in E_n$ and $x \in E \setminus E_n$, respectively. So by monotonicity of integration

$$\int_E |f_n|^p - \int_E |f|^p + p\int_E |f|^{p-1}f(f - f_n) \ge c\int_{E_n} |f_n - f|^p + c\int_{E\setminus E_n} |f_n - f|^2|f|^{p-2}.$$

As above, the left hand side approaches 0, and so the right hand side approaches 0. So for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ where for all $n > N$ we have

$$c\int_{E_n} |f_n - f|^p < \frac{c\epsilon}{2} \text{ and } c\int_{E\setminus E_n} (f_n - f)^2|f|^{p-2} < \frac{c\epsilon^2}{4\|f\|_p^p}.$$

Then for all $n > N$ we have

$$\begin{aligned} \frac{\epsilon}{2} &> \left\{\|f\|_p^p\right\}^{1/2}\left\{\int_{E\setminus E_n} (f_n - f)^2|f|^{p-1}\right\}^{1/2} \\ &\ge \left\{\int_{E\setminus E_n} |f|^p\right\}^{1/2}\left\{\int_{E\setminus E_n} (f_n - f)^2|f|^{p-1}\right\}^{1/2} \\ &\ge \left\{\int_{E\setminus E_n} (|f|^{p/2})^2\right\}^{1/2}\left\{\int_{E\setminus E_n} (|f_n - f||f|^{(p-2)/2})^2\right\}^{1/2} \end{aligned}$$

$$\geq \int_{E\setminus E_n} |f|^{p/2} \cdot |f_n - f||f|^{(p-2)/2} \text{ by the Cauchy-Schwarz Inequality (or Hölder with } p = q = 2)$$
$$= \int_{E\setminus E_n} |f|^{p-1}|f_n - f|$$
$$\geq \int_{E\setminus E_n} |f_n - f|^{p-1}|f_n - f| \text{ since } |f| \geq |f_n - f| \text{ on } E \setminus E_n$$
$$= \int_{E\setminus E_n} |f_n - f|^p.$$

Therefore
$$\int_E |f_n - f|^p = \int_{E_n} |f_n - f|^p + \int_{E\setminus E_n} |f_n - f| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
So $\|f_n - f\|_p \to 0$ as $n \to \infty$ and $\{f_n\} \to f$ with respect to the L^p norm for $1 \leq p < 2$. ∎

Note. The Radon-Riesz Theorem does not hold for $p = 1$. Let $n \in \mathbb{N}$ and define $f_n(x) = 1+\sin(nx)$ on $I = [-\pi, \pi]$. then $\{f_n\} \rightharpoonup f$ in $L^1[\pi, \pi]$ where $f \equiv 1$ by Royden and Fitzpatrick's Theorem 8.11 since

$$\begin{aligned}\lim_{n\to\infty}\left(\int_{-\pi}^{x} f_n\right) &= \lim_{n\to\infty}\int_{-\pi}^{x}(1+\sin(nt)) = \lim_{n\to\infty}\left(t - \frac{1}{n}\cos(nt)\right)\Big|_{-\pi}^{\pi} \\ &= \lim_{n\to\infty}\left((x - \frac{1}{n}\cos(nx)) - (\pi - \frac{1}{n}\cos(n\pi))\right) = x + \pi = \int_{-\pi}^{\pi} 1.\end{aligned}$$

Also, as above,
$$\lim_{n\to\infty}\|f_n\|_1 = \lim_{n\to\infty}\int_{-\pi}^{\pi}|1+\sin(nt)|\,dt = 2\pi = \|f\|_1.$$
However, $\{f_n\} = \{1 + \sin(nx)\}$ does not converge to $f \equiv 1$ in $L^1[-\pi, \pi]$ since

$$\begin{aligned}\lim_{n\to\infty}\|f_n - f\|_1 &= \lim_{n\to\infty}\left(\int_{-\pi}^{\pi}|\sin(nx)|\,dx\right) = \lim_{n\to\infty}\left(2n\int_0^{\pi/n}\sin(nx)\,dx\right) \\ &= \lim_{n\to\infty} 2n\left(-\frac{1}{n}\cos(nx)\right)\Big|_0^{\pi/n} = \lim_{n\to\infty}(-2\cos\pi + 2\cos 0) = 4 \neq 0.\end{aligned}$$

So $\{f_n\} \rightharpoonup f$ in $L^1(I)$, $\lim_{n\to\infty}\|f_n\|_1 = \|f\|_1$, but $\{f_n\}$ does not converge strongly to f in $L^1(I)$.

Who is the Riesz guy? Frigyes Riesz (1880–1956) was born in Hungary, studied in Zurich, Budapest, and Göttingen. In 1911 he started working at the University of Kolozsvàr, which was moved to Szeged in 1920. In 1946 he started at the University of Budapest. We first encountered F. Riesz in the *Riesz-Fischer Theorem* of Section 7.3. This result was found independently by Riesz and Emil Fischer (not to be confused with the better known mathematician Ronald A. Fisher) and states that the L^p spaces are complete. We have also seen the Riesz Representation Theorem in Section 8.1, which classifies bounded linear functionals on L^p. This result is from 1909 and is probably Riesz's best known result. Riesz is the one to introduce the idea of weak sequential convergence in L^p. The index of Royden and Fitzpatrick also mentions Riesz's Lemma, Riesz's Theorem, the Riesz-Fréchet Representation Theorem, the Riesz-Markov Theorem, and the Riesz-Schauder Theorem. Riesz is one of the founders of functional analysis and he published *LeÇons d'analyse fonctionnelle* in 1952, which he coathored with Béla Szőkefnalvy-Nagy. This book was translated into English by L. Boron and published as *Functional Analysis* in 1955 (this is the reference used for the above proof).

Frigyes Riesz (1880–1956)

References:

1. Mathematics Genealogy Project: `genealogy.math.ndsu.nodak.edu/`

2. The MacTutor History of Mathematics archive: `www-history.mcs.st-and.ac.uk/Biographies/Riesz.html`

3. "Riesz, Frigyes (Frédéric)." Complete Dictionary of Scientific Biography (2008) by I. Grattan-Guiness: `www.encyclopedia.com`

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