Real Analysis, Chapter 2 Study Guide

Chapter 2. Lebesgue Measure

The following is a *brief* list of topics covered in Chapter 2 of Royden and Fitzpatrick's *Real Analysis*, 4th edition, and the associated supplements. This list is not meant to be comprehensive, but only gives a list of several important topics.

2.1. Introduction.

The four desired properties of a measure, finite additivity, monotonicity, countable subadditivity.

2.2. Lebesgue Outer Measure.

Outer measure, the outer measure of an interval is its length (Proposition 2.1), outer measure is translation invariant (Proposition 2.2), outer measure is countably subadditive (Proposition 2.3).

2.3. The σ-algebra of Lebesgue Measureable Sets.

Lebesgue measurable set and the Carathéodory splitting condition, inner measure, the definition of "Lebesgue measurable set" in terms of inner and outer measure, measure zero sets are measurable (Proposition 2.4), the union of a finite collection of measurable sets is measurable (Proposition 2.5), Proposition 2.6 (which implies finite additivity of Lebesgue measure), the union of a countable collection of measurable sets is measurable (Proposition 2.7), Lebesgue measure is countably additive (Proposition 2.13), intervals are measurable (Proposition 2.8), the translate of a measurable set is measurable (Proposition 2.10).

2.4. Outer and Inner Approximation of Lebesgue Measurable Sets.

Conditions equivalent to a set being measurable (Theorem 2.11), the inner and outer approximation of a measurable set, "nearly."

2.5. Countable Additivity, Continuity, and the Borel-Cantelli Lemma.

The definition of Lebesgue measure, measure is continuous (Theorem 2.15), almost everywhere, The Borel-Cantelli Lemma.

Supplement: Dr. Bob's Axiom of Choice Centennial Lecture.

The seven axioms of Zermelo, the Axiom of Choice, the Weak-Ordering Principle (Zermelo's Theorem), Maximal Principle I (Zorn's Lemma), Maximal Principle II (Tukey's Lemma), Well-Orderings, chains, upper bound of a chain, maximal element of a chain.

2.6. Nonmeasurable Sets (4th Ed.).

Translation of bounded measurable set by countably infinite bounded set of real numbers (Lemma 2.6), rationally equivalent, choice set, the Vitali construction of a nonmeasurable set (Theorem 2.17).

2.6. Nonmeasurable Sets (3rd Ed.).

"Circle plus," measure is "circle plus invariant" (Lemma A), rationally equivalent, choice function, the Axiom of Choice, set P, Theorem B, set P is not measurable, the use of the Axiom of Choice in the construction of set P and its meaning.

Supplement. Nonmeasurable Sets and the Banach-Tarski Paradox.

Shifting to infinity, the Hilbert hotel, "doubling the unit interval" based on the Vitalli set P, the Hausdorff Paradox, the Banach-Tarski Paradox, R. M. Robertson's result (the paradox can be accomplished with five pieces).

2.7. The Cantor Set and the Cantor Lebesgue Function.

The Cantor set $\mathbf{C}$ and its complement $\mathcal{O}$ in $[0,1]$, the Cantor set is closed/uncountable/measure zero (Proposition 2.19), fat Cantor set (Exercise 2.39), the Cantor-Lebesgue function, properties of the Cantor set (Proposition 2.20), there is a measurable set that is a subset of the Cantor set which is not Borel (Proposition 2.22).

Supplement. The Cardinality of the Set of Lebesgue Measurable Sets.

Robert Solovay's result (the Axiom of Choice is necessary for the construction of a nonmeasurable set), definition of the Cantor Set, the cardinality of the Cantor Set, $|\mathcal{M}| = \aleph_2$ (Theorem A), $|\mathcal{P}(\mathbb{R}) \setminus \mathcal{M}| = \aleph_2$ (Theorem B).