

*General Relativity* by Robert M. Wald

## Chapter 2: Manifolds and Tensor Fields

### 2.1. Manifolds

**Note.** An *event* is a point in spacetime. In prerelativity physics and in special relativity, the space of all events is  $\mathbb{R}^4$ . In general relativity, we will keep the idea that spacetime is locally “like”  $\mathbb{R}^4$ , but allow spacetime to have geometric properties different from  $\mathbb{R}^4$  (that is, we do not require spacetime to be flat). This is analogous to the fact that the Earth is locally like  $\mathbb{R}^2$ , but is globally different from  $\mathbb{R}^2$ .

**Note.** When making an analogy with the surface of the Earth, we take advantage of the fact that a sphere is embedded in 3-space (an *extrinsic* property). When considering all of spacetime, it would not make sense to think of how it is embedded in a higher dimensional space (for this would imply something “outside” of the universe, and such things are beyond science — if not beyond meaning!). We must therefore study spacetime from within (that is, we can only study *intrinsic* properties of spacetime). For such a study, we need to develop the abstract idea of an  $n$ -manifold.

**Definition.** Let  $x = (x^1, x^2, \dots, x^n)$  and  $y = (y^1, y^2, \dots, y^n)$  be points in  $\mathbb{R}^n$ . Define the (Euclidean) *distance* between  $x$  and  $y$  as

$$|x - y| = \left[ \sum_{\mu=1}^n (x^\mu - y^\mu)^2 \right]^{1/2}.$$

The *open ball* in  $\mathbb{R}^n$  of center  $y$  and radius  $r$  is

$$\{x \in \mathbb{R}^n \mid |x - y| < r\}.$$

An *open set* in  $\mathbb{R}^n$  is any set which can be expressed as an arbitrary union of open balls.

**Note.** In fact, an open set in  $\mathbb{R}^n$  can be expressed as a countable collection of open balls. This is called the *Lindelöf Property* of  $\mathbb{R}^n$ .

**Definition.** A function  $\varphi : X \rightarrow Y$  is *one-to-one* if for distinct  $x^1, x^2 \in X$  we have  $\varphi(x^1) \neq \varphi(x^2)$ . (If a function  $\varphi$  is one-to-one, then we can define its inverse  $\varphi^{-1}$ .) The function  $\varphi$  is *onto* if for each  $y \in Y$ , there exists  $x \in X$  such that  $\varphi(x) = y$ . If  $\varphi : X \rightarrow \mathbb{R}^n$  (where  $X$  is any set) then  $\varphi$  is  $C^\infty$  if  $\varphi$  is infinitely differentiable. (Notice that  $\varphi$  is a vector valued function and should be treated as an  $n$ -tuple of scalar valued functions  $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^n)$ . Saying  $\varphi$  is infinitely differentiable is equivalent to saying that  $\varphi^\mu$  is infinitely differentiable for each  $\mu$ . It does not make sense to talk about the differentiability of a function between arbitrary *sets* — we need more structure than that.)

**Note.** In the opinion of your humble instructor, a manifold is best thought of in terms of paper mache. As opposed to taking small pieces of paper (which represent little open sets from a 2-dimensional vector space) and soaking them in water and glue (allowing us to bend and warp the pieces; that is, to continuously transform them), we consider small pieces of  $n$ -dimensional space which are mapped continuously to open subsets of the manifold. Instead of pasting the pieces of paper onto a wire frame and overlapping them, we require that the continuous mappings compose to give a certain level of differentiability. More precisely, we have the following.

**Definition.** An  $n$ -dimensional,  $C^\infty$ , real manifold is a set of  $M$  points together with a collection  $\{O_\alpha\}$  of subsets of  $M$  such that

1.  $\bigcup_{\alpha} O_\alpha = M$ .
2. For each  $\alpha$ , there is a one-to-one and onto map  $\psi_\alpha : O_\alpha \rightarrow U_\alpha$  where  $U_\alpha$  is an open subset of  $\mathbb{R}^n$ .
3. For any two  $O_\alpha, O_\beta \subset M$  with  $O_\alpha \cap O_\beta \neq \emptyset$ . We have the sets  $\psi_\alpha[O_\alpha \cap O_\beta] \subset \mathbb{R}^n$  and  $\psi_\beta[O_\alpha \cap O_\beta] \subset \mathbb{R}^n$  open and the function  $\psi_\beta \circ \psi_\alpha^{-1}$  is  $C^\infty$ .

Notice that  $\psi_\beta \circ \psi_\alpha^{-1}$  maps  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , so differentiability is defined. Each map  $\psi_\alpha$  is a *coordinate system* (as physicists say; mathematicians call it a “chart”).

**Convention.** For a manifold  $M$ , we require that the cover  $\{O_\alpha\}$  and the set of coordinate systems  $\{\psi_\alpha\}$  are maximal. That is, all coordinate systems compatible with parts 2 and 3 of the definition of a manifold are included. This avoids the complication of defining new manifolds from given manifolds by simply adding or deleting a coordinate system.

**Note.** We say  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $C^\infty$  if all partials (pure and mixed) of all orders exist and are continuous.

**Definition.** The maps  $\psi_\alpha$  above which associate elements of  $\mathbb{R}^n$  with points of the manifold are called *charts* or *coordinate systems*.

**Note.** The function  $\psi_\beta \circ \psi_\alpha^{-1}$  maps  $\mathbb{R}^n$  to  $\mathbb{R}^n$  and  $\psi_\alpha[O_\alpha \cap O_\beta] \subset U_\alpha$  and  $\psi_\beta[O_\alpha \cap O_\beta] \subset U_\beta$ . We have:

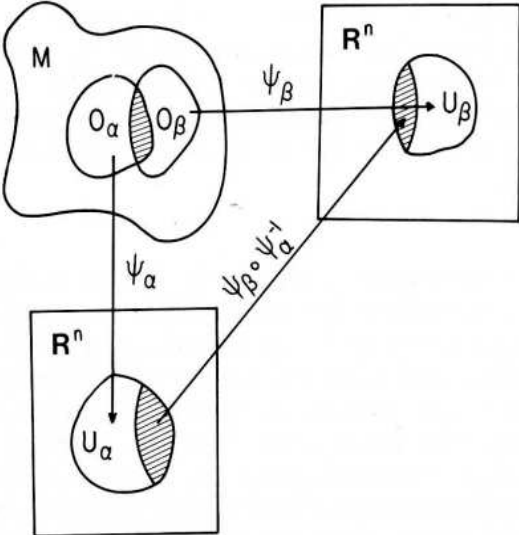


Figure 2.1 from Wald, page 13.

**Example.**  $\mathbb{R}^n$  is an  $n$ -manifold (trivially).

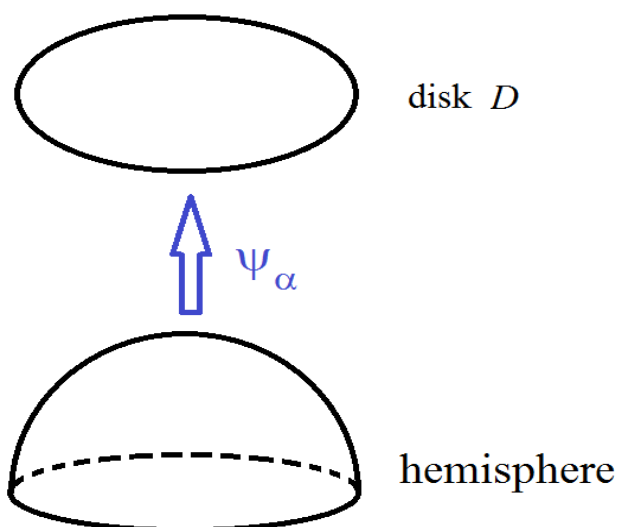
**Example. (Problem 2.1(a))** The 2-sphere

$$S^2 = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$$

is a 2-manifold. First, we cannot simply map  $S^2$  continuously onto  $\mathbb{R}^2$  since  $S^2$  is a compact subset of  $\mathbb{R}^3$  and  $\mathbb{R}^2$  is not a compact set of  $\mathbb{R}^3$  (and continuous functions map compact sets onto compact sets). Therefore we have to cover  $S^2$  with sets  $\{O_\alpha\}$  and map these sets into  $\mathbb{R}^2$ . So define the six hemispherical open sets  $O_i^\pm$  for  $i = 1, 2, 3$ :

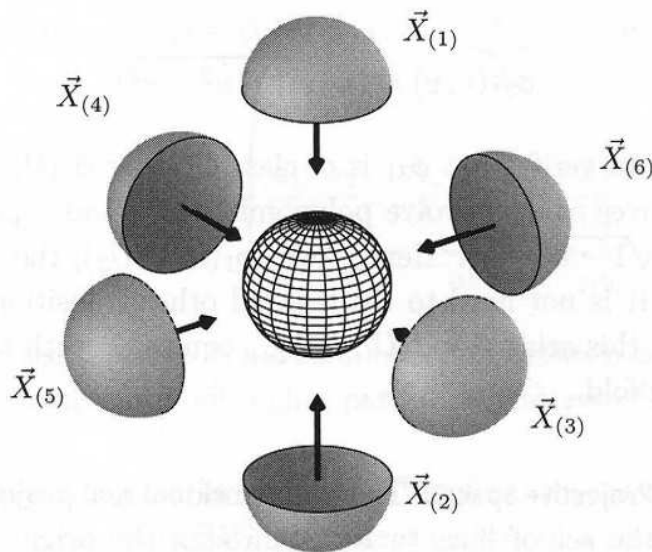
$$O_I^\pm = \{(x^1, x^2, x^3) \in S^2 \mid \pm x^i > 0\}$$

these correspond to the top half, bottom half, right half, left half, front half, and bottom half of the sphere). Then *Property 1 of the definition of manifold is satisfied*:  $\bigcup_{\alpha} O_\alpha = S^2 = M$ , Next, take  $U_\alpha = D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$  for  $\alpha = 1, 2, \dots, 6$ . We define the six  $\psi_\alpha$  by projecting the hemisphere onto  $D$ :



Each of the six (open) hemispheres are mapped onto the (open) unit disk.

That is,  $f_1^\pm(x^1, x^2, x^3) = (x^2, x^3)$ ,  $f_2^\pm(x^1, x^2, x^3) = (x^1, x^3)$ , and  $f_3^\pm(x^1, x^2, x^3) = (x^1, x^2)$ . Then the coordinate system  $\{\psi_\alpha\} = \{f_i^\pm\}$  are one-to-one and onto maps mapping the  $O_\alpha$ 's onto open set  $D$ . Hence *property 2 of the definition of manifold is satisfied*. (The following is Problem 2.1a.) Now  $(f_j^\pm)^{-1}$  maps  $D$  onto one of the six hemispheres. So  $(f_i^\pm) \circ (f_j^\pm)^{-1}$  maps  $D$  onto a (not necessarily nonempty) subset of  $D$ . In particular,  $(f_i^+) \circ (f_i^-)^{-1}$  and  $(f_i^-) \circ (f_i^+)^{-1}$ , for  $i = 1, 2, 3$  (so this covers 6 cases) are nowhere defined and Property 3 does not apply. (For example,  $(f_3^+)^{-1}$  maps  $D$  onto the upper (open) hemisphere of  $S^2$  but  $f_3^-$  is only defined on the lower [open] hemisphere of  $S^2$  so  $(f_3^-) \circ (f_3^+)^{-1}$  has an empty domain.) Next, if  $i \neq j$  then  $(f_i^\pm) \circ (f_j^\pm)^{-1}$  maps  $D$  onto one of the following (depending on  $i$  and  $j$ ):



From *Differential Geometry of Manifolds*, by Stephen Lovett (A. K. Peters, 2010).

Without loss of generality, consider  $(f_2^+) \circ (f_1^+)^{-1}$ . Then  $(f_1^+)^{-1}$  maps  $D$  onto the “front” of  $S^2$  and  $f_2^+$  then maps the intersection of the “front” and “right” of  $S^2$  onto the right side of  $D$ .

So

$$(x, y) \xrightarrow{(f_1^+)^{-1}} (\sqrt{1-x^2-y^2}, x, y) \xrightarrow{f_2^+} (\sqrt{1-x^2-y^2}, y).$$

For  $(x, y) \in D$ , this composition is  $C^\infty$  (notice that the  $+$  need not correspond to the  $+$ , and so this covers  $4 \times 6 = 24$  cases). This leaves 6 of the  $6 \times 6 = 36$  cases to consider, namely  $(f_i^+) \circ (f_i^+)^{-1}$  and  $(f_i^-) \circ (f_i^-)^{-1}$  for  $i = 1, 2, 3$ . In each of these 6 cases, the composition is the identity function from  $D$  to  $D$  and so is  $C^\infty$ . Therefore, *Property 3 of the definition of manifold is satisfied* and hence  $S^2$  is a manifold.

**Note.** Analogous to the previous example, we can show that the  $n$ -sphere  $S^n$  is an  $n$ -manifold.

**Definition.** Given two manifolds  $M$  and  $M'$  of dimensions  $n$  and  $n'$ , respectively, we define the *product manifold*  $M \times M'$ . We take as the points of  $M \times M'$  the collection of all pairs  $(p, p')$  where  $p \in M$  and  $p' \in M'$ . For the collection of subsets of  $M \times M'$  we take the collection of all  $\{O_{\alpha\beta} = O_\alpha \times O_\beta\}$  where  $O_\alpha$  is a subset of  $M$  and  $O_\beta$  is a subset of  $M'$  (where  $O_\alpha$  and  $O_\beta$  are as described in the definition of manifold). Finally, we define the coordinate system  $\{\varphi_{\alpha\beta}\}$  where  $\varphi_{\alpha\beta} : O_{\alpha\beta} \rightarrow U_{\alpha\beta} = U_\alpha \times U_\beta \subset \mathbb{R}^{n+n'}$  as  $\varphi_{\alpha\beta}(p, p') = (\psi_\alpha(p), \psi'_\beta(p'))$  where  $p \in O_\alpha$ ,  $p' \in O_\beta$ ,  $\psi_\alpha : O_\alpha \rightarrow U_\alpha$ , and  $\psi'_\beta : O'_\beta \rightarrow U'_\beta$ .

**Note.** Certainly  $\{O_{\alpha\beta}\}$  above covers  $M \times M'$ . Also, for all  $\alpha\beta$ , the function  $\varphi_{\alpha\beta}$  maps  $O_{\alpha\beta}$  one-to-one and onto an open set of  $\mathbb{R}^{n+n'}$  (namely,  $U_{\alpha\beta} = U_\alpha \times U'_\beta$ ).

**Note.** We now can define differentiability of a function from one manifold  $M$  to another  $M'$ . We do so by using the charts of the manifolds. Let  $M$  and  $M'$  have chart maps  $\{\psi_\alpha\}$  and  $\{\psi'_\beta\}$ , respectively, and let  $f : M \rightarrow M'$ . Then we have

$$\mathbb{R}^n \xrightarrow{\psi_\alpha^{-1}} M \xrightarrow{f} M' \xrightarrow{\psi'_\beta} \mathbb{R}^{n'},$$

and  $\psi'_\beta \circ f \circ \psi_\alpha^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$ . So we use this mapping to define differentiability of  $f$ .

**Definition.** Let  $M$  and  $M'$  be manifolds with chart maps  $\{\psi_\alpha\}$  and  $\{\psi'_\beta\}$ , respectively. We have  $f : M \rightarrow M'$  is  $C^\infty$  if for each  $\alpha$  and  $\beta$  the map  $\psi'_\beta \circ f \circ \psi_\alpha^{-1}$  taking  $U_\alpha \subset \mathbb{R}^n$  into  $U'_\beta \subset \mathbb{R}^{n'}$  is  $C^\infty$ . If  $f : M \rightarrow M'$  is  $C^\infty$ , one to one, onto and has a  $C^\infty$  inverse, then  $f$  is a *diffeomorphism* and  $M$  and  $M'$  are said to be *diffeomorphic*.

**Note.** Diffeomorphic manifolds have identical manifold structure.

**Note.** These notes are based in part on the fall 2011 Honors in Discipline project of Jessie Deering-Jamieson titled “What is a Manifold?”

*Last updated:* June 19, 2016