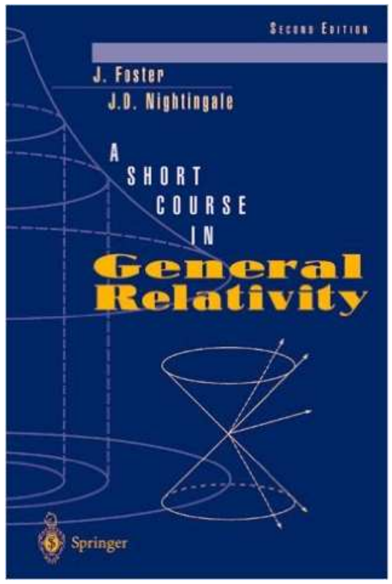
Special Topic: Black Holes

Primary Source: *A Short Course in General Relativity*, 2nd Ed., J. Foster and J.D. Nightingale, Springer-Verlag, N.Y., 1995.



Note. Suppose we have an isolated spherically symmetric mass M with radius r_B which is at rest at the origin of our coordinate system. Then we have seen that the solution to the field equations in this situation is the *Schwarzschild solution*:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\varphi^2 - r^2 \sin^2 \varphi \, d\theta.$$

Notice that at $r = 2M$, the metric coefficient g_{11} is undefined. Therefore this solution is only valid for $r > 2M$. Also, the solution was derived for points outside of the mass, and so r must be greater than the radius of the mass r_B. Therefore the Schwarzschild solution is only valid for $r > \max\{2M, r_B\}$.

Definition. For a spherically symmetric mass M as above, the value $r_S = 2M$ is the *Schwarzschild radius* of the mass. If the radius of the mass is less than the Schwarzschild radius (i.e. $r_B < r_S$) then the object is called a *black hole.*

Note. In terms of "traditional" units, $r_S = 2GM/c^2$. For the Sun, $r_S = 2.95$ km and for the Earth, $r_S = 8.86$ mm.

Note. Since the coordinates (t, r, φ, θ) are inadequate for $r \le r_S$, we introduce a new coordinate which will give metric coefficients which are valid for all r. In this way, we can explore what happens inside of a black hole!

Note. We keep r, φ, and θ but replace t with

$$v = t + r + 2M \ln \left| \frac{r}{2M} - 1 \right|. \qquad (*)$$

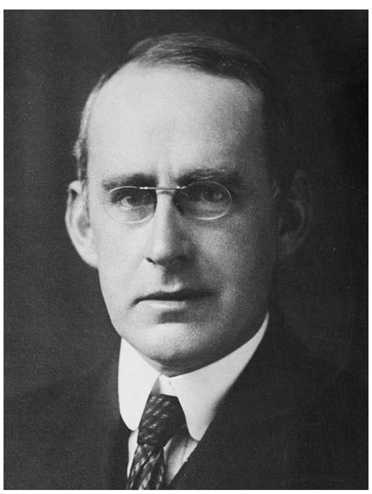
Theorem. In terms of (v, r, φ, θ), the Schwarzschild solution is

$$d\tau^2 = (1 - 2M/r)dv^2 - 2dv\,dr - r^2 d\varphi^2 - r^2 \sin^2 \varphi\, d\theta. \qquad (**)$$

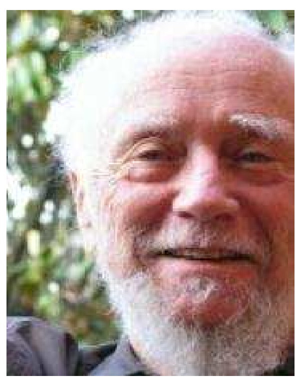
These new coordinates are the *Eddington-Finkelstein coordinates.*

Proof. Homework! (Calculate dt^2 in terms of dv and dr, then substitute into the Schwarzschild solution.)

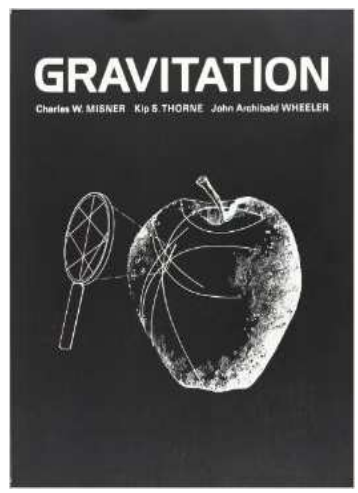
Note. These coordinates are named for Arthur Eddington (1882–1944) and David Finkelstein (1929–2016), even though neither ever wrote down these coordinates or the metric in these coordinates. Eddington's paper mentions the Schwarzschild solution and appear as "A Comparison of Whitehead's and Einstein's Formulae," *Nature*, **113**(2832), 192 (1924) (it is available online as `www.strangepaths.com/files/eddington.pdf`). Finkelstein's work is "Past-Future Asymmetry of the Gravitational Field of a Point Particle," *Physical Reviews*, **110**, 965–967 (1958). Roger Penrose (in "Gravitational Collapse and Space-Time Singularities," *Physical Review Letters*, **14**(3), 57–59 (1965)) seems to have been the first to write down the form we use, but credits it (wrongly) to the above paper by Finkelstein, and, in his Adams Prize essay later that year, to Eddington and Finkelstein. Most influentially, Misner, Thorne and Wheeler, in their colossal book *Gravitation* (W. H. Freeman and Company, 1973), refer to the coordinates as "Eddington-Finkelstein coordinates." This information is from the Wikipedia webpage at `en.wikipedia.org/wiki/Eddington%E2%80%93Finkelstein_coordinates` (accessed 7/7/2016).



Arthur Eddington



David Finkelstein



Gravitation

Eddington image from Wikipedia, Finkelstein image from: `www.physics.gatech.edu/user/david-finkelstein`, and *Gravitation* image from Amazon.

Note. Each of the coefficients of $d\tau^2$ in Eddington-Finkelstein coordinates is defined for all nonzero $r > r_B$. Therefore we can explore what happens for $r < r_S$ in a black hole. We are particularly interested in light cones.

Note. Let's consider what happens to photons emitted at a given distance from the center of a black hole. We will ignore φ and θ and take $d\varphi = d\theta = 0$. We want to study the radial path that photons follow (i.e. radial lightlike geodesics). Therefore we consider $d\tau = 0$. Then $(**)$ implies

$$(1 - 2M/r)dv^2 - 2dv\,dr = 0,$$

$$(1 - 2M/r)\frac{dv^2}{dr^2} - 2\frac{dv}{dr} = 0,$$

$$\left(\frac{dv}{dr}\right)\left((1 - 2M/r)\frac{dv}{dr} - 2\right) = 0.$$

Therefore we have a lightlike geodesic if $\dfrac{dv}{dr} = 0$ or if $\dfrac{dv}{dr} = \dfrac{2}{1 - 2M/r}$.

Note. First, let's consider radial lightlike geodesics for $r > r_S$. Differentiating $(*)$ gives

$$\begin{aligned} \frac{dv}{dr} &= \frac{dt}{dr} + 1 + \frac{1}{r/2M - 1} \\ &= \frac{dt}{dr} + \frac{r/2M}{r/2M - 1} = \frac{dt}{dr} + \frac{1}{1 - 2M/r}. \end{aligned}$$

With the solution $dv/dr = 0$, we find

$$\frac{dt}{dr} = \frac{-1}{1 - 2M/r}.$$

Notice that this implies that $dt/dr < 0$ for $r > 2M$. Therefore for $dv/dr = 0$ we see that as time (t) increases, distance from the origin (r) decreases. Therefore $dv/dr = 0$ gives the *ingoing lightlike geodesics*. With the solution $dv/dr = 2/(1-2M/r)$, we find

$$\frac{dt}{dr} = \frac{2}{1-2M/r} - \frac{1}{1-2M/r} = \frac{1}{1-2M/r}.$$

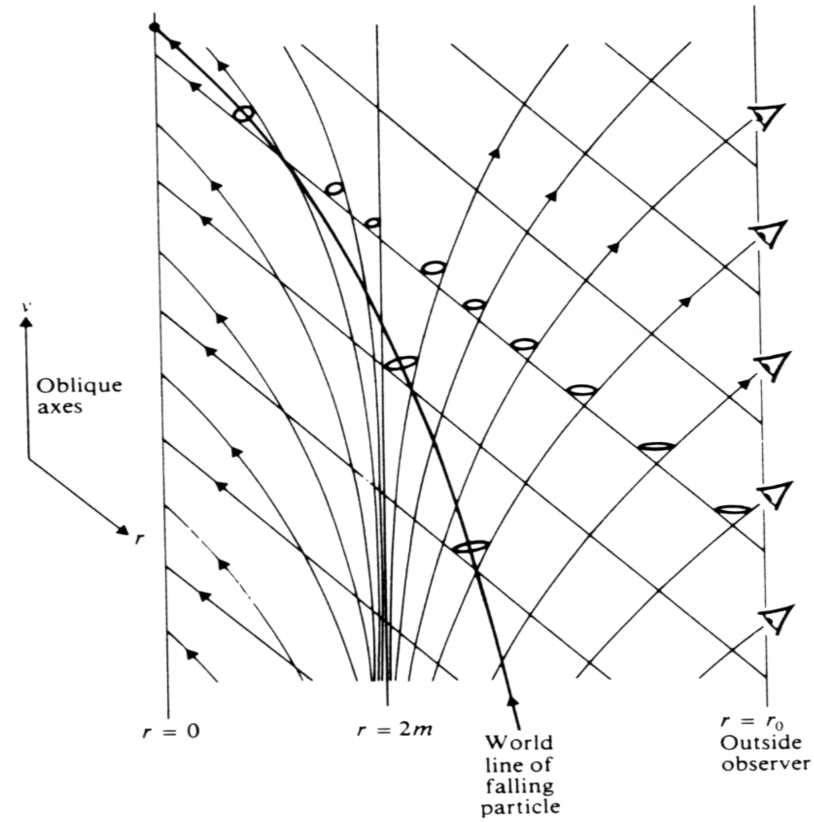
Notice that this implies that $dt/dr > 0$ for $r > 2M$. Therefore for $dv/dr = 2/(1-2M/r)$, we see that as time (t) increases, distance from the origin (r) increases. Therefore $dv/dr = 2/(1-2M/r)$ gives the *outgoing lightlike geodesics*. Therefore for $r > 2M$, a flash of light at position r will result in photons that go towards the black hole and photons that go away from the black hole (remember, we are only considering radial motion).

Note. Now, let's integrate the two solutions above so that we can follow lightlike geodesics in (r, v) "oblique coordinates." Integrating the solution $dv/dr = 0$ gives $v = A$ (A constant). Integrating the solution $dv/dr = 2/(1-2M/r)$ gives

$$v = \int \frac{dv}{dr} dr = \int \frac{2\,dr}{1-2M/r} = \int \frac{2r\,dr}{r-2M}$$

$$= 2\int \left(1 + \frac{2M}{r-2M}\right) dr = 2r + 4M\ln|r-2M| + B,$$

B constant. In the following figure (Figure 4.13 from the Foster and Nightingale book), we use oblique axes and choose A such that $v = A$ gives a line 45° to the horizontal (as in flat spacetime). The choice of B just corresponds to a vertical shift in the graph of $v = 2r + 4M\ln|r-2M|$ and does not change the shape of the graph (so we can take $B = 0$).



The little circles represent small local lightcones. Notice that a photon emitted towards the center of the black hole will travel to the center of the black hole (or at least to r_B). A photon emitted away from the center of the black hole will escape the black hole if it is emitted at $r > r_S = 2M$. However, such photons are "pulled" towards $r = 0$ if they are emitted at $r < r_S$. Therefore, any light emitted at $r < r_S$ will not escape the black hole and therefore cannot be seen by an observer located at $r > r_S$. Thus the name *black* hole. Similarly, an observer outside of the black hole cannot see any events that occur in $r \leq r_S$ and the sphere $r = r_S$ is called the *event horizon* of the black hole.

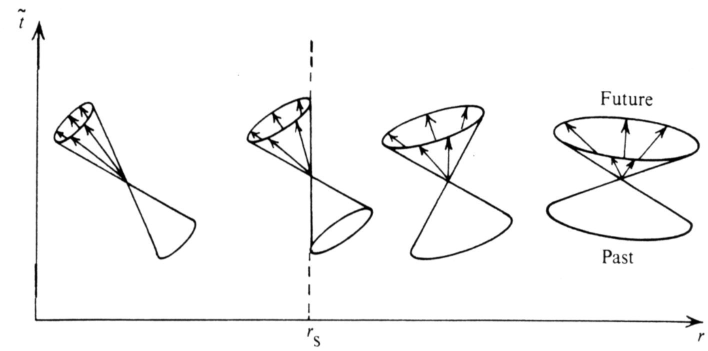
Note. Notice the worldline of a particle which falls into the black hole. If it periodically releases a flash of light, then the outside observer will see the time between the flashes taking a longer and longer amount of time. There will therefore be a *gravitational redshift* of photons emitted near $r = r_S$ ($r > r_S$). Also notice that the outside observer will see the falling particle take longer and longer to reach $r = r_S$. Therefore the outside observer sees this particle fall towards $r = r_S$, but the particle appears to move slower and slower.

Note. Notice that the particle falling into the black hole will see photons from the outside observer. You can trace the path of a photon along a line of slope 1 backwards from the time the particle reaches $r = 0$ to determine the last thing the particle sees before it reaches the singularity at the center of the black hole. You may have heard it said that when you fall into a black hole you will see the whole evolution of the universe to $t = \infty$ before you reach the singularity. In fact, Neil deGrasse Tyson comments in the 2014 *Cosmos, A Spacetime Odyssey*, "A Sky Full of Ghosts" (episode 4) that "If you somehow survive the perilous journey across the event horizon, you'd be able to look back and see the entire future history of the universe unfold before your eyes." (This is at 34:00 in the version without commercials.) But this contradicts our diagram. Our diagram, though, is for a *Schwarzschild black hole*! The details which explain deGrasse Tyson's comment can be found on the University of California, Riverside webpage at:

`http://math.ucr.edu/home/baez/physics/Relativity/`
`BlackHoles/fall_in.html`

It explains that for charged or rotating black holes (assumptions which we did not make in the Schwarzschild black hole model) "timelike wormholes" serve as gateways to other universes and instead of hitting a singularity at the center of the black hole, you experience an infinite speed-up effect as you enter the wormhole. In this setting, you would see the "entire future history of the universe" (deGrasse Tyson goes on to discuss wormholes in his presentation).

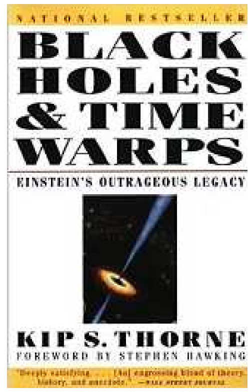
Note. Notice that the radial lightlike geodesics determined by $dv/dr = 2(1-2M/r)$ have an asymptote at $r = r_S$. This will result in lightcones tilting over towards the black hole as we approach $r = r_S$:



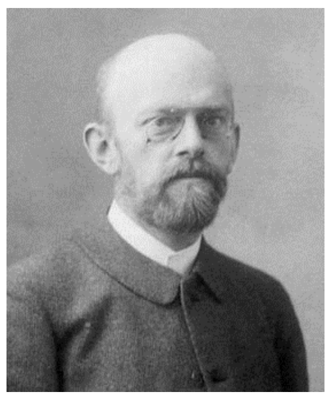
(Figure 46, page 93 of *Principles of Cosmology and Gravitation*, M. Berry, Cambridge University Press, 1976.) Again, far from the black hole, light cones are as they appear in flat spacetime. For $r \approx r_S$ and $r > r_S$, light cones tilt over towards the black hole, but photons can still escape the black hole. At $r = r_S$, photons are either trapped at $r = r_S$ (those emitted radially to the black hole) or are drawn into the black hole. For $r < r_S$, all worldlines are directed towards $r = 0$. Therefore *anything* inside r_S will be drawn to $r = 0$. All matter in a black hole is therefore concentrated at $r = 0$ in a *singularity* of infinite density.

Note. The Schwarzschild solution is an *exact* solution to the field equations. Such solutions are rare, and sometimes are not appreciated in their fullness when introduced. Here is a brief history from *Black Holes and Time Warps: Einstein's*

Outrageous Legacy by Kip Thorne (W.W. Norton and Company, 1994).



1915 Einstein (and David Hilbert (1862-1943)) formulate the field equations (which Einstein published in 1916).



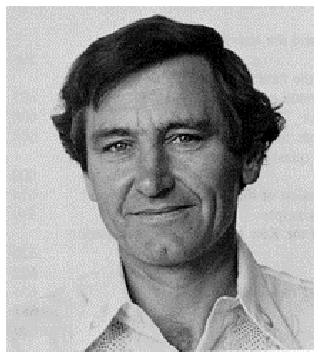
David Hilbert in 1910 (from the MacTutor History of Mathematics archive).

1916 Karl Schwarzschild presents his solution which later will describe nonspinning, uncharged black holes.

1916 & 1918 Hans Reissner and Gunnar Nordström give their solutions, which later will describe nonspinning, charged black holes. (The ideas of black holes, white dwarfs, and neutron stars did not become part of astrophysics until the 1930's, so these early solutions to the field equations were not intended to address any questions involving black holes.)

1958 David Finkelstein introduces a new reference frame for the Schwarzschild solution, resolving the 1939 Oppenheimer-Snyder paradox in which an imploding star freezes at the critical (Schwarzschild) radius as seen from outside, but implodes through the critical radius as seen from inside.

1963 Roy Kerr (1931–) gives his solution to the field equations.

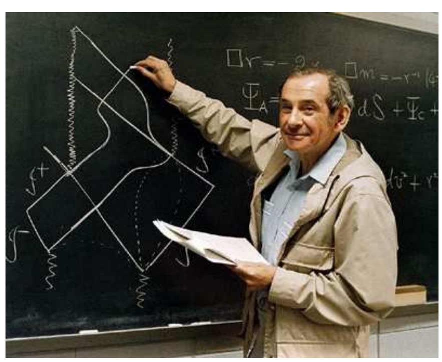


From `http://scienceworld.wolfram.com/biography/KerrRoy.html`

1965 Boyer and Lindquist, Carter, and Penrose discover that Kerr's solution describes a spinning black hole.

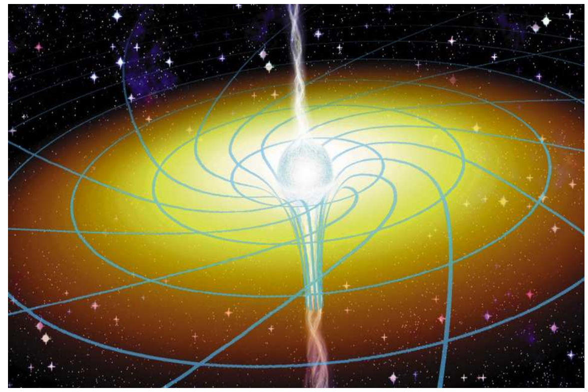
Some other highlights include:

1967 Werner Israel (1931–) proves rigorously the first piece of the black hole "no hair" conjecture: a nonspinning black hole must be precisely spherical.



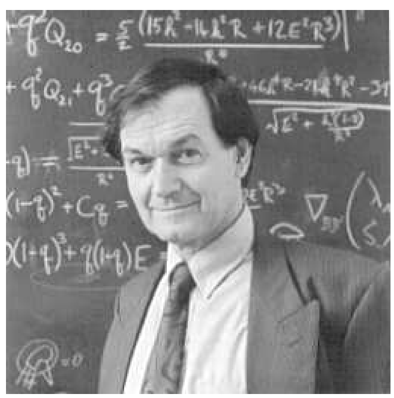
From `http://www.thecanadianencyclopedia.ca/en/article/werner-israel/`

1968 Brandon Carter (1942–) uses the Kerr solution to show frame dragging around a spinning black hole.



From `luth2.obspm.fr/~luthier/carter/` and `www.wbur.org/npr/136000666/nasa-mission-confirms-two-of-einsteins-theories`

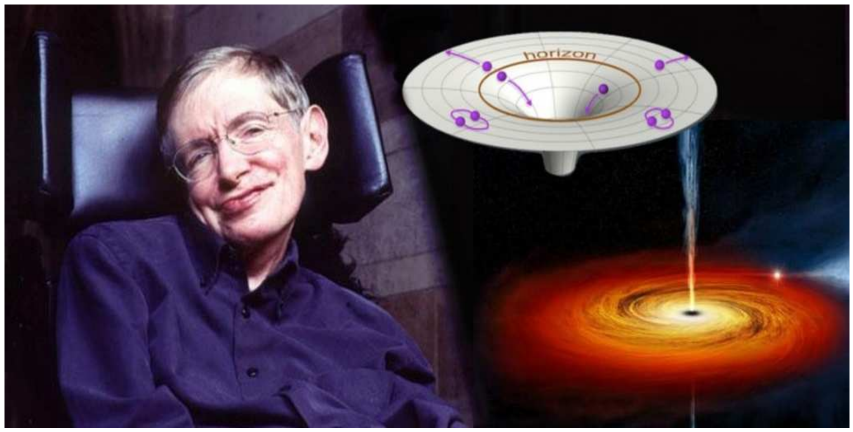
1969 Roger Penrose (1931–) describes how the rotational energy of a black hole can be extracted in what became known as the Penrose mechanism.



From `http://www.worldofescher.com/misc/penrose.html`

1972 Carter, Hawking, and Israel prove the "no hair" conjecture for spinning black holes. The implication of the no hair theorem is that a black hole is described by three parameters: mass, rotational rate, and charge.

1974 Stephen Hawking (1942–2018) shows that it is possible to associate a temperature and entropy with a black hole. He uses quantum theory to show that black holes can radiate (the so-called *Hawking radiation*).

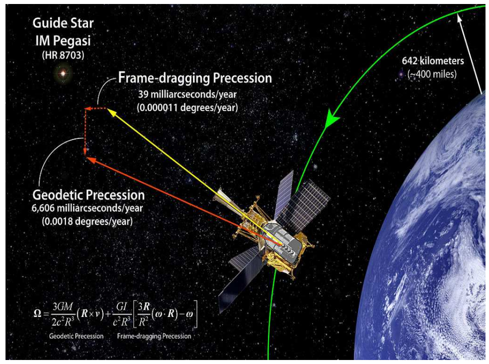


From `http://www.universetoday.com/123487/who-is-stephen-hawking/`

1993 Russell Hulse (1950–) and Joseph Taylor (1941–) are awarded the Nobel Prize for an indirect detection of gravitational waves from a binary pulsar.

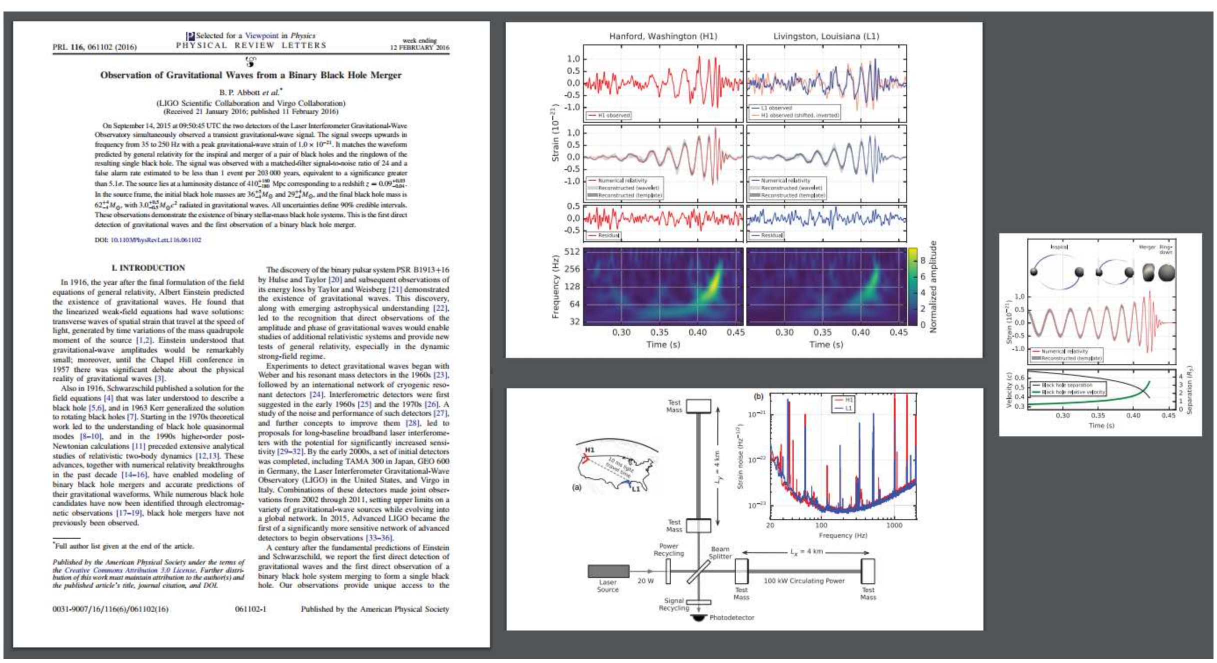
From `http://physicscapsule.com/editorial/ligo-gravitational-waves/`

2011 NASA's Gravity Probe B confirms frame dragging around the Earth.



From `https://en.wikipedia.org/wiki/Gravity_Probe_B`

2015 Gravitational waves were directly detected for the first time by the Laser Interferometer Gravity Wave Observatory (LIGO). The detection was of the merger of a 29 solar mass black hole with a 36 solar mass black hole. Three solar masses were converted into energy and released in the form of gravitational waves. The merger occurred 1.3 billion light years away in the direction of the Magellanic Clouds.

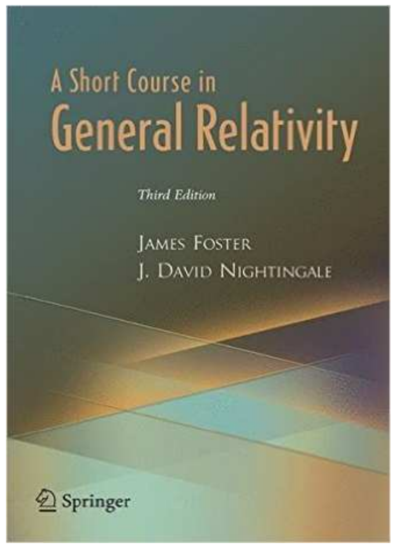


These results appeared in Abbott *et al.* "Observation of Gravitational Waves from a Binary Black Hole Merger," *Physical Review Letters* **116**, 061102 (2016). The paper is online at: `journals.aps.org/prl/pdf/10.1103/PhysRevLett.116.061102` .

2019 The Event Horizon Telescope collaboration is a global system of radio telescopes. During 2017 radio telescopes at eight sites made observations of the supermassive black hole at the center of the galaxy M87. On April 10, 2019 it was announced at six news conferences held around the world that this black hole had been imaged. A non-technical article describing the project is on the JPL website "How Scientists Captured the First Image of a Black Hole" at `https://www.jpl.nasa.gov/edu/news/2019/4/19/how-scientists-captured-the-first-image-of-a-black-hole/`. A technical discussion is given in "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole" published in *The Astrophysical Journal Letters* 875:L1 2019 April 10 (available online in PDF at `https://iopscience.iop.org/article/10.3847/2041-8213/ab0ec7/pdf`). The image from the JPL website is:

Note. Where to go next…It would be logical to explore the Kerr solution for a rotating black hole next. The two references I recommend are:

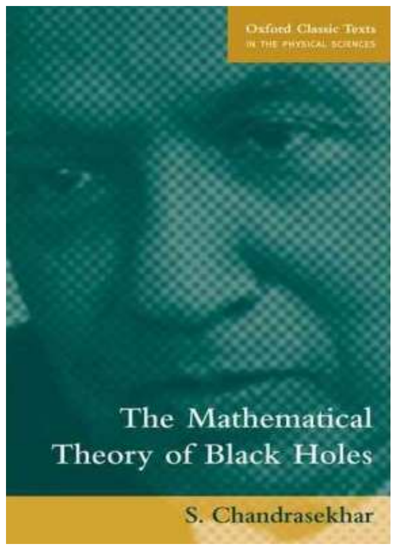
1. James Foster and J. David Nightingale, *A Short Course in General Relativity*, 3rd Edition, NY: Springer-Verlag (2010). A section in this new edition is "4.10. Rotating objects; the Kerr solution."



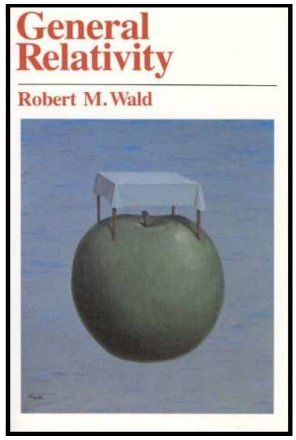
2. Subrahmanyan Chandrasekhar (1910-1995), *The Mathematical Theory of Black Holes*, Oxford University Press (1983) (reprinted in paperback 1992). This includes several chapters on the Kerr solution: "Chapter 6, The Kerr Metric," "Chapter 7. The Geodesics in the Kerr Space-Time," "Chapter 8. Electromagnetic Waves in Kerr Geometry," "Chapter 9. The Gravitational Perturbations of the Kerr Black-Hole," and "Chapter 10. Spin-$\frac{1}{2}$ Particles in Kerr Geometry." It also includes chapters on the Schwarzschild solution (Chapter 3), perturbations of the Schwarzschild black hole (Chapter 4), and the Reisnner-Nordström solution (Chapter 5). Chandrasekhar states the Kerr metric as (see page 289):

$$ds^2 = \rho^2 \frac{\Delta}{\Sigma^2}(dt)^2 - \frac{\Sigma^2}{\rho^2}\left(d\varphi - \frac{2aMr}{\Sigma^2}dt\right)^2 \sin^2\theta - \frac{\rho^2}{\Delta}(dr)^2 - \rho^2(d\theta)^2$$

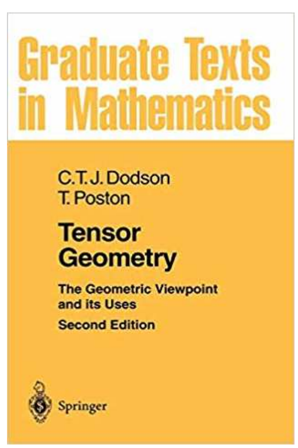
where $\Delta = r^2 - 2Mr + a^2$ (see equation (45) and page 280), $\rho^2 = r^2 + a^2\cos^2\theta$ (see equation (112)), $\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$ (see equations (51) and (121)), and a is a constant (see page 279).



Note. For a deeper study of differential geometry and general relativity, I recommend Robert M. Wald's *General Relativity*, University of Chicago Press (1984). This is the one book which, in my opinion, strikes the best balance between a pure math approach (which can be a bit esoteric and lose sight of the underlying physics) and a primarily physical approach (which can lose sight of the mathematical details)! It *is* a physics book, though! The first Part (Chapters 1–6) introduces manifolds, tensors, curvature, the field equations, and the Schwarzschild solution; This would be the standard material covered in a one semester course, according to Wald. The second Part (Chapters 7–14) concerns solution techniques and a chapter on black holes (including the topics of charged Kerr black holes, the Penrose mechanism for the extraction of energy from a black hole, and the thermodynamics of black holes). There are appendices on topological spaces, differential forms, Lie derivatives, and Killing fields.



Note. For a deeper study from a pure math book, I like C.T.J. Dodson and T. Poston's *Tensor Geometry: The Geometric Viewpoint and Its Uses*, Springer Verlag, Graduate Texts in Mathematics #130 (1991). This book seems to do a good job on the necessary math, including chapters on affine spaces, dual spaces, metric vector spaces (this is where the metric tensor is defined), manifolds, differentiation, geodesics, and curvature. There is also a chapter on special relativity and a chapter on general relativity (but these two chapters seem a bit conversant and light). I have some online notes for this book at: `http://faculty.etsu.edu/gardnerr/5310/notes-Dodson-Poston.htm`.



Revised: 7/4/2019