1.7 Geodesics

Note. A curve $\vec{\alpha}(s)$ on a surface M can curve in two different ways. First, $\vec{\alpha}$ can bend *along with* surface M (the "normal curvature" discussed above). Second, $\vec{\alpha}$ can bend *within* the surface M (the "geodesic curvature" to be defined).

Recall. For curve $\vec{\alpha}$ on surface M, α'' can be written as components tangent and normal to M as $\vec{\alpha}'' = \vec{\alpha}''_{\text{tan}} + \alpha''_{\text{nor}}$ where

$$\begin{gathered}\vec{\alpha}''_{\text{tan}} = (u^{r\prime\prime} + \Gamma^r_{ij} u^{i\prime} u^{j\prime})\vec{X}_r \\ \vec{\alpha}''_{\text{nor}} = (L_{ij} u^{i\prime} u^{j\prime})\vec{U}\end{gathered} \tag{29}$$

and the parameters on the right hand side are defined in Section 5. $\vec{\alpha}''_{\text{nor}}$ reflects the curvature of $\vec{\alpha}$ due to the bending of M and $\vec{\alpha}''_{\text{tan}}$ reflects the curvature of $\vec{\alpha}$ within M. Now

$$\vec{\alpha}''_{\text{tan}} \cdot \vec{U} = \left\{(u^{r\prime\prime} + \Gamma^r_{ij} u^{i\prime} u^{j\prime})\vec{X}_r\right\} \cdot \vec{U} = 0$$

(recall $\vec{U} = \dfrac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X} \times \vec{X}_2\|}$) and

$$\begin{aligned}\vec{\alpha}''_{\text{tan}} \cdot \vec{\alpha}' &= \vec{\alpha}''_{\text{tan}} \cdot \vec{\alpha}' + 0 = \vec{\alpha}''_{\text{tan}} \cdot \vec{\alpha}' + \vec{\alpha}''_{\text{nor}} \cdot \vec{\alpha}' \\ &\qquad (\text{recall } \vec{\alpha}' = u^{i\prime}\vec{X}_i \text{ and } \vec{X}_i \cdot \vec{U} = 0) \\ &= (\vec{\alpha}''_{\text{tan}} + \vec{\alpha}''_{\text{nor}}) \cdot \vec{\alpha}' = \vec{\alpha}'' \cdot \vec{\alpha}' = 0 \\ &\qquad (\text{recall } \|\vec{\alpha}'\| = \|\vec{\alpha}'(s)\| = 1 \text{ and } \prime = d/ds).\end{aligned}$$

Therefore $\vec{\alpha}''_{\text{tan}}$ is orthogonal to both $\vec{U}$ and $\vec{\alpha}'$. If we define $\vec{w}$ as the unit vector $\vec{w} = \vec{U} \times \vec{\alpha}'$, then $\vec{\alpha}''_{\text{tan}}$ is a multiple of $\vec{w}$ (and $\vec{w}$ is a vector tangent to M).

Definition I-7. Let $\vec{\alpha}(s)$ be a curve on M where s is arc length. The *geodesic curvature* of $\vec{\alpha}$ at $\vec{\alpha}(s)$ is the function $k_g = k_g(s)$ defined by

$$\vec{\alpha}''_{\text{tan}} = k_g\vec{w} = k_g(\vec{U} \times \vec{\alpha}'). \tag{30}$$

Recall. The scalar triple product of three vectors (in $\mathbb{R}^3$) satisfies:

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B}.$$

Theorem 1.7.A. The geodesic curvature k_g of curve $\vec{\alpha}$ in surface M can be calculated as

$$k_g = \vec{U} \cdot \vec{\alpha}' \times \vec{\alpha}''. \tag{31}$$

Proof. Since $k_g\vec{w} = \vec{\alpha}''_{\text{tan}}$ we have

$$k_g\vec{w} \cdot \vec{w} = \vec{\alpha}''_{\text{tan}} \cdot \vec{w} = \vec{\alpha}''_{\text{tan}} \cdot (\vec{U} \times \vec{\alpha}')$$

or

$$\begin{aligned} k_g &= (\vec{\alpha}''_{\text{tan}} + \vec{\alpha}''_{\text{nor}}) \cdot \vec{U} \times \vec{\alpha}' \\ &\quad \text{(since } \vec{\alpha}''_{\text{nor}} \text{ is parallel to } \vec{U}\text{)} \\ &= \vec{\alpha}'' \cdot \vec{U} \times \vec{\alpha}' = \vec{U} \cdot \vec{\alpha}' \times \vec{\alpha}''. \end{aligned}$$

∎

Definition I-8. Let $\vec{\alpha} = \vec{\alpha}(s)$ be a curve on a surface M. Then $\vec{\alpha}$ is a *geodesic* if $\vec{\alpha}''_{\text{tan}} = \vec{0}$ (or equivalently, if $\vec{\alpha}'' = \vec{\alpha}''_{\text{nor}}$) at every point of $\vec{\alpha}$.

Note. A geodesic on a surface is, in a sense, as "straight" as a curve can be on the surface. That is, $\vec{\alpha}$ has no curvature within the surface. For example, on a sphere the geodesics are great circles.

Note. If $\vec{\alpha}$ is a geodesic on M if and only if

$$u^{r\prime\prime} + \Gamma^r_{ij} u^{i\prime} u^{j\prime} = 0 \text{ for } r = 1, 2 \qquad (32a)$$

and

$$\vec{U} \cdot \vec{\alpha}' \times \vec{\alpha}'' = 0. \qquad (32b)$$

(We'll use these LOTS!)

Example (Exercise 1.7.4(a)). Prove that on a surface of revolution, every meridian is a geodesic.

Proof. Suppose

$$\vec{X}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

Let $\vec{m}(s) = (f(s)\cos v, f(s)\sin v, g(s))$ be a meridian of the surface (where we assume the curve has been parameterized in terms of arclength s). Then

$$\begin{aligned}
\vec{m}'(s) &= (f'(s)\cos v, f'(s)\sin v, g'(s)) \\
\vec{m}'' &= (f''(s)\cos v, f''(s)\sin v, g''(s)) \\
\vec{m}' \times \vec{m}'' &= ((f'(s)g''(s) - f''(s)g'(s))\sin v, (-f'(s)g''(s) + f''(s)g'(s))\cos v, 0).
\end{aligned}$$

Now

$$\vec{X}_1 \times \vec{X}_2 = (-f(s)g'(s)\cos v, -f(s)g'(s)\sin v, f'(s)f(s))$$

and so

$$\vec{U} = \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|} = \frac{(-f(s)g'(s)\cos v, -f(s)g'(s)\sin v, f'(s)f(s))}{(f(s)g'(s))^2 + (f'(s)f(s))^2}.$$

Therefore

$$\begin{aligned}
\vec{U} \cdot \vec{m}' \times \vec{m}'' &= \frac{1}{(f(s))^2\{(g'(s))^2 + (f'(s))^2\}} \\
&\quad \times\{(f'(s)g''(s) - f''(s)g'(s))(-f(s)g'(s))\cos v \sin v \\
&\quad +(-f'(s)g''(s) + f''(s)g'(s))(-f(s)g'(s))\cos v \sin v, 0)\} \\
&= \frac{1}{(f(s))^2\{(g'(s))^2 + (f'(s))^2\}}(0) = 0.
\end{aligned}$$

Therefore $\vec{m}(s)$ is a geodesic (see equation (32b)). ∎

Definition. Let $\vec{X}(u^1, u^2)$ be a surface and let g_{ij} (see page 34) and Γ^r_{ij} (see page 43) be as defined in Sections 4 and 5. The *Christoffel symbols of the first kind* are

$$\Gamma_{ijk} = \Gamma^r_{ij} g_{rk} \qquad (33)$$

for $i, j, k = 1, 2$.

Definition. The Γ^r_{ij} defined in Section 1.5 are the *Christoffel symbols of the second kind.*

Note. Since $\Gamma^r_{ij} = \Gamma^r_{ji}$ (see (17), page 43) then $\Gamma_{ijk} = \Gamma_{jik}$; that is, there is symmetry in the first two indices of the Christoffel symbols of the first kind. Also, since $(g_{ij})^{-1} = (g^{ij})$, we have $\Gamma^m_{ij} = \Gamma_{ijk} g^{km}$.

Theorem 1.7.B. Let $\vec{X}(u^1, u^2)$ be a surface and let g_{ij} and Γ^r_{ij} be as defined in Sections 4 and 5. Then

$$\Gamma_{ijk} = \vec{X}_{ij} \cdot \vec{X}_k \tag{34}$$

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \tag{36}$$

and

$$\Gamma^r_{ij} = \frac{1}{2} g^{kr}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right). \tag{37}$$

Proof. Since $\vec{X}_{ij} = \Gamma^r_{ij}\vec{X}_r + L_{ij}\vec{U}$ (by definition, see page 43) then

$$\vec{X}_{ij} \cdot \vec{X}_k = \Gamma^r_{ij}\vec{X}_r \cdot \vec{X}_k + (L_{ij}\vec{U}) \cdot \vec{X}_k = \Gamma^r_{ij} g_{rk} + 0 = \Gamma_{ijk}$$

establishing the first identity (recall $g_{rk} = \vec{X}_r \cdot \vec{X}_k$). Next,

$$\frac{\partial g_{ik}}{\partial u^j} = \frac{\partial}{\partial u^j}[\vec{X}_i \cdot \vec{X}_k] = \vec{X}_{ij} \cdot \vec{X}_k + \vec{X}_{kj} \cdot \vec{X}_i = \Gamma_{ijk} + \Gamma_{kji}. \tag{35a}$$

Permuting the indices:

$$\frac{\partial g_{ji}}{\partial u^k} = \Gamma_{jki} + \Gamma_{ikj} \quad \text{and} \quad \frac{\partial g_{kj}}{\partial u^i} = \Gamma_{kij} + \Gamma_{jik}. \tag{35b and 35c}$$

Now

$$\begin{aligned}
\Gamma_{ijk} &= \frac{1}{2}(2\Gamma_{ijk}) = \frac{1}{2}(\Gamma_{ijk} + \Gamma_{jik}) \\
&= \frac{1}{2}(\Gamma_{ijk} + \Gamma_{kji} - \Gamma_{kji} + \Gamma_{kij} - \Gamma_{kij} + \Gamma_{jik}) \text{ since } \Gamma_{ijk} = \Gamma_{jik} \text{ by} \\
&\quad \text{symmetry in the first two indices} \\
&= \frac{1}{2}\{(\Gamma_{ijk} + \Gamma_{kji}) + (\Gamma_{kij} + \Gamma_{jik}) - (\Gamma_{jki} + \Gamma_{ikj})\} \text{ since } \Gamma_{kji} = \Gamma_{jki} \text{ and} \\
&\quad \Gamma_{kij} = \Gamma_{ijk} \text{ by symmetry in the first two indices} \\
&= \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right)
\end{aligned}$$

and the second identity is established. Finally, multiplying this identity on both sides by g^{kr}, summing over k and using the definition of Γ^r_{ij} we have

$$\Gamma^r_{ij} = \Gamma_{ijk}g^{kr} = \frac{1}{2}g^{kr}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right)$$

(recall $(g^{ij}) = (g_{ij})^{-1}$), and the third identity is established. ∎

Note. Since the Christoffel symbols depend only on the metric form (or First Fundamental Form), they are part of the intrinsic geometry of the surface M.

Definition. Let $\vec{X}(u^1, u^2)$ be a surface. Then the coordinates $\vec{X}_1$ and $\vec{X}_2$ are *orthogonal* if $g_{12} = g_{21} = 0$. (This makes sense since $g_{ij} = \vec{X}_i \cdot \vec{X}_j$.)

Corollary 1.7.A. Let $\vec{X}(u^1, u^2)$ be a surface and let g_{ij} and Γ^r_{ij} be as defined in Sections 4 and 5. If $\vec{X}_1$ and $\vec{X}_2$ are orthogonal coordinates, then

$$\Gamma^r_{ij} = \frac{1}{2g_{rr}}\left(\frac{\partial g_{ir}}{\partial u^j} + \frac{\partial g_{jr}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^r}\right)$$

(no sums over any of i, j, r).

Proof. Since $g_{12} = g_{21} = 0$, then $g^{12} = g^{21} = 0$ and $g^{11} = 1/g_{11}$, $g^{22} = 1/g_{22}$. The result follows from the above theorem. ∎

Corollary 1.7.B. With the hypotheses of the previous corollary (with $i, j, r = 1, 2$), when $j = r$

$$\Gamma^r_{ir} = \frac{1}{2g_{rr}}\frac{\partial g_{rr}}{\partial u^i} = \frac{1}{2}\frac{\partial}{\partial u^i}[\ln g_{rr}]$$

and when $i = j \neq r$

$$\Gamma^r_{ii} = \frac{1}{2g_{rr}}\left(-\frac{\partial g_{ii}}{\partial u^r}\right).$$

Proof. Follows from $g_{12} = g_{21} = 0$. ∎

Note. By symmetry, $\Gamma^r_{ij} = \Gamma^r_{ji}$, and so the previous two corollaries cover all possible cases of orthogonal coordinates when $i, j, r \in \{1, 2\}$ (i.e., when we deal with two dimensions). In dimensions 3 and greater (in particular, in the 4 dimensional spacetime of Chapter III) we have a third case which we state now, and address in detail later:

Theorem 1.7.C. In dimensions 3 and greater, if coordinates are mutually orthogonal, then for i, j, r all distinct, $\Gamma^r_{ij} = 0$. (In the event that one or more of i, j, r are equal, the above corollaries apply.)

Note. In the case of orthogonal coordinates, if we return to Gauss' notation:

$$g_{11} = E, \qquad g_{12} = g_{21} = F = 0, \qquad g_{22} = G$$

we have the First Fundamental Form (or metric form) $ds^2 = Edu^2 + Gdv^2$ on surface $\vec{X}(u, v)$. In this notation, the Christoffel symbols are then

$$\begin{array}{rl} \Gamma^1_{11} = \frac{E_u}{2E} & \Gamma^2_{22} = \frac{G_v}{2G} \\ \Gamma^1_{12} = \Gamma^1_{21} = \frac{E_v}{2E} & \Gamma^2_{21} = \Gamma^2_{12} = \frac{G_u}{2G} \\ \Gamma^1_{22} = -\frac{G_u}{2E} & \Gamma^2_{11} = -\frac{E_v}{2G}. \end{array} \qquad (40)$$

Example 17, page 62. In the Euclidean plane, $ds^2 = du^2 + dv^2$. Therefore $E = G = 1$ and all the Christoffel symbols are 0. Therefore a geodesic $\vec{\alpha}$ satisfies

$$u^{r\prime\prime} + \Gamma^r_{ij} u^{i\prime} u^{j\prime} = 0$$

for $r = 1, 2$, or $u^{r\prime\prime} = 0$ for $r = 1, 2$. That is, $u^{1\prime\prime} = u'' = 0$ and $u^{2\prime\prime} = v'' = 0$. Therefore $u(s) = as + b$ and $v(s) = cs + d$ for some a, b, c, d. Therefore, geodesics in the Euclidean plane are straight lines.

Note. We will show in Theorem I-9 that the shortest path on a surface joining two points is a geodesic. This theorem, combined with the previous example PROVES that the shortest distance between two points in a plane is a straight line. Oddly enough, you've probably never seen this PROVED before!

Example 18, page 62. Consider a sphere of radius r with "geographic coordinates" (like latitude and longitude) u and v. Then the sphere is given by

$$\vec{X}(u, v) = (r \cos u \cos v, r \sin u \cos v, r \sin v)$$

(see Example 7, page 23). The metric form is (see page 33) $ds^2 = r^2 \cos^2 v du^2 + r^2 dv^2$ (since there is no $du\, dv$ term, $F = g_{12} = g_{21} = 0$ and these coordinates are orthogonal). Therefore $E = r^2 \cos^2 v$ and $G = r^2$ (a constant). Then $E_u = G_u = G_v = 0$ and the nonzero Christoffel symbols are

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{E_v}{2E} = \frac{-2r^2 \cos v \sin v}{2r^2 \cos^2 v} = -\tan v$$

$$\Gamma^2_{11} = \frac{-E_v}{2G} = \frac{2r^2 \cos v \sin v}{2r^2} = \cos v \sin v.$$

It is shown in Exercise 1.7.14 (at the end of this section) that this implies geodesics are great circles.

Note. In Example 19 page 62, it is shown that the Euclidean plane when equipped with polar coordinates (which are orthogonal coordinates) yields geodesics which are lines (as expected).

Note. In general, to determine the geodesics for a surface, requires that one solve differential equations. This can be difficult (sometimes impossible to do in terms of elementary functions). In Chapter III we will compute some geodesics in 4-dimensional spacetime (in fact, planets and light follow geodesics if 4-D spacetime).

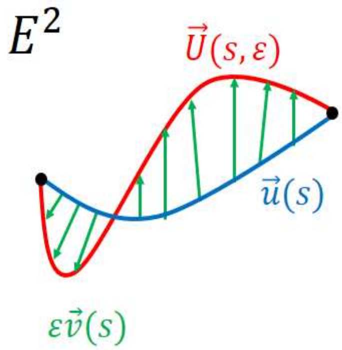
Theorem I-9. Let $\vec{\alpha}(s)$, $s \in [a, b]$ be a curve on the surface $M : \vec{X}(u^1, u^2)$, where s is arclength. If $\vec{\alpha}$ is the shortest possible curve on M connecting its two end points, then $\vec{\alpha}$ is a geodesic.

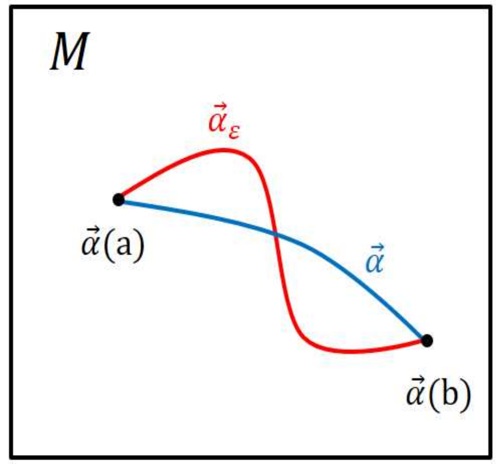
Idea of Proof. We will vary $\vec{\alpha}(s)$ by a slight amount ε. Then comparing the arclength of $\vec{\alpha}$ from $\vec{\alpha}(a)$ to $\vec{\alpha}(b)$ with the arclength of the slightly varied curve from $\vec{\alpha}(a)$ to $\vec{\alpha}(b)$ and assuming $\vec{\alpha}$ to yield the minimal arclength, we will show that $\vec{\alpha}$ satisfies equation (32a) and is therefore a geodesic.

Proof. Let $\vec{\alpha}(s) = \vec{X}(u^1(s), u^2(s))$. Consider the family of curves of the form

$$U^i(s, \varepsilon) = u^i(s) + \varepsilon v^i(s)$$

for $i = 1, 2$, $s \in [a, b]$ where v^i are smooth functions with $v^i(a) = v^i(b) = 0$ for $i = 1, 2$ (so $\vec{X}(U^1, U^2)$ still joins $\vec{\alpha}(a)$ and $\vec{\alpha}(b)$), $\vec{X}(U^1, U^2) \subset M$, but otherwise v^i are arbitrary. We take $\vec{\alpha}_\varepsilon(s) = \vec{X}(U^1(s, \varepsilon), U^2(s, \varepsilon))$.





Let $L(\varepsilon)$ denote the length of $\vec{\alpha}_\varepsilon$:

$$L(\varepsilon) = \int_a^b \lambda(s,\varepsilon)\, ds$$

where

$$\lambda(s,\varepsilon) = \left\{ g_{ij}(U^1,U^2)\frac{\partial U^i}{\partial s}\frac{\partial U^j}{\partial s}\right\}^{1/2}$$

(the square root of the metric form of M along $\vec{\alpha}_\varepsilon$). Now L has a minimum at $\varepsilon = 0$ so

$$\frac{d}{d\varepsilon}[L(\varepsilon)] = \frac{d}{d\varepsilon}\left[\int_a^b \lambda(s,\varepsilon)\, ds\right] = \int_a^b \frac{\partial}{\partial\varepsilon}[\lambda(s,\varepsilon)]\, ds$$

(since λ and $\partial\lambda/\partial\varepsilon$ are continuous) satisfies

$$L'(0) = \int_a^b \frac{\partial}{\partial\varepsilon}[\lambda(s,0)]\, ds = 0.$$

Now

$$\begin{aligned}
\frac{\partial\lambda}{\partial\varepsilon} &= \frac{\partial}{\partial\varepsilon}\left[\left(g_{ij}(U^1,U^2)\frac{\partial U^i}{\partial s}\frac{\partial U^j}{\partial s}\right)^{1/2}\right] \\
&= \frac{1}{2}(\lambda(s,\varepsilon))^{-1}\left\{\frac{\partial}{\partial\varepsilon}[g_{ij}(U^1,U^2)]\frac{\partial U^i}{\partial s}\frac{\partial U^j}{\partial s} + g_{ij}(U^1,U^2)\frac{\partial}{\partial\varepsilon}\left[\frac{\partial U^i}{\partial s}\right]\frac{\partial U^j}{\partial s}\right.
\end{aligned}$$

$$
\begin{aligned}
&+g_{ij}(U^1,U^2)\frac{\partial U^i}{\partial s}\frac{\partial}{\partial \varepsilon}\left[\frac{\partial U^j}{\partial s}\right]\Bigg\} \\
=\ & \frac{1}{2\lambda(s,\varepsilon)}\left\{\left(\frac{\partial}{\partial U^1}[g_{ij}(U^1,U^2)]\frac{\partial U^1}{\partial \varepsilon}+\frac{\partial}{\partial U^2}[g_{ij}(U^1,U^2)]\frac{\partial U^2}{\partial \varepsilon}\right)\frac{\partial U^i}{\partial s}\frac{\partial U^j}{\partial s}\right. \\
&\left.+2g_{ij}(U^1,U^2)\frac{\partial U^i}{\partial s}\frac{\partial}{\partial \varepsilon}\left[\frac{\partial U^j}{\partial s}\right]\right\} \\
=\ & \frac{1}{2\lambda(s,\varepsilon)}\left\{\left(\frac{\partial}{\partial U^k}[g_{ij}(U^1,U^2)]\frac{\partial U^k}{\partial \varepsilon}\right)\frac{\partial U^i}{\partial s}\frac{\partial U^j}{\partial s}+2g_{ij}(U^1,U^2)\frac{\partial U^i}{\partial s}\frac{\partial^2 U^j}{\partial \varepsilon\,\partial s}\right\} \\
=\ & \frac{1}{2\lambda(s,\varepsilon)}\left\{\left(\frac{\partial g_{ij}}{\partial U^k}v^k\right)\frac{\partial U^i}{\partial s}\frac{\partial U^j}{\partial s}+2g_{ij}\frac{\partial U^i}{\partial s}\frac{\partial^2 U^j}{\partial \varepsilon\,\partial s}\right\}\ \text{(notice that we sum} \\
&\text{over } k \text{ here since we treat the partial derivative with respect to } U^k \\
&\text{as if it were a subscript)}
\end{aligned}
$$

since $\dfrac{\partial U^k}{\partial \varepsilon} = v^k$. With $\varepsilon = 0$, $\dfrac{\partial U^j}{\partial \varepsilon} = v^j$ and $\lambda(s,0) = 1$ (because $\lambda(s,\varepsilon)\,ds|_{\varepsilon=0} = \|\alpha'\|\,ds = 1\,ds$; s is arclength on $\vec{\alpha} = \vec{\alpha}_0$) we have

$$\frac{\partial \lambda}{\partial \varepsilon}(s,0) = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial U^k}v^k U^{i\prime}U^{j\prime} + 2g_{ik}U^{i\prime}v^{k\prime}\right)$$

and since $\varepsilon = 0$ implies $U^i = u^i$, then

$$\frac{\partial \lambda}{\partial \varepsilon}(s,0) = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial u^k}v^k u^{i\prime}u^{j\prime} + 2g_{ik}u^{i\prime}v^{k\prime}\right)$$

and so

$$L'(0) = \frac{1}{2}\int_a^b \left(\frac{\partial g_{ij}}{\partial u^k}u^{i\prime}u^{j\prime}v^k + 2g_{ik}u^{i\prime}v^{k\prime}\right)ds = 0.$$

Now by Integration by Parts

$$
\begin{aligned}
&\int_a^b 2g_{ik}u^{i\prime}v^{k\prime}ds \qquad && \text{Let } u = 2g_{ik}u^{i\prime} \text{ and } dv = v^{k\prime}ds. \\
& && \text{Then } du = \frac{\partial}{\partial s}[2g_{ik}u^{i\prime}]ds \text{ and } v = \int v^{k\prime}(s)ds = v^k. \\
&= \left\{2g_{ik}u^{i\prime}v^k - \int \frac{\partial}{\partial s}[2g_{ik}u^{i\prime}]v^k ds\right\}\Bigg|_a^b
\end{aligned}
$$

$$= \; 0 - \int_a^b \frac{\partial}{\partial s}[2g_{ik}u^{i\prime}]v^k ds \text{ since } v^k(a) = v^k(b) = 0.$$

Therefore

$$\begin{aligned} L'(0) &= \frac{1}{2}\int_a^b \left(\frac{\partial g_{ij}}{\partial u^k}u^{i\prime}u^{j\prime}v^k - \frac{\partial}{\partial s}[2g_{ik}u^{i\prime}]v^k\right) ds \\ &= \frac{1}{2}\int_a^b \left(\frac{\partial g_{ij}}{\partial u^k}u^{i\prime}u^{j\prime} - \frac{\partial}{\partial s}[2g_{ik}u^{i\prime}]\right) v^k ds \\ &= 0. \end{aligned}$$

Since the integral must be zero *for all arbitrary* v^k (and since $\int_a^b f(s)g(s)\,ds = 0$ for arbitrary $g(s)$ implies $f(s) = 0$), then the remaining part of the integrand must be zero:

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial u^k}u^{i\prime}u^{j\prime} - \frac{\partial}{\partial s}[g_{ik}u^{i\prime}] = 0$$

for $k = 1, 2$. Now when $\varepsilon = 0$, $U^i = u^i$ and

$$\begin{aligned} \frac{\partial}{\partial s}[g_{ik}u^{i\prime}] &= \frac{\partial}{\partial s}[g_{ik}(u^1, u^2)u^{i\prime}] \\ &= \left(\frac{\partial g_{ik}}{\partial u^1}\frac{du^1}{ds} + \frac{\partial g_{ik}}{\partial u^2}\frac{du^2}{ds}\right) u^{i\prime} + g_{ik}(u^1, u^2)\frac{du^{i\prime}}{ds} \\ &= \left(\frac{\partial g_{ik}}{\partial u^1}u^{1\prime} + \frac{\partial g_{ik}}{\partial u^2}u^{2\prime}\right) u^{i\prime} + g_{ik}(u^1, u^2)u^{i\prime\prime} \\ &= \left(\frac{\partial g_{ik}}{\partial u^j}u^{j\prime}\right) u^{i\prime} + g_{mk}u^{m\prime\prime} = \frac{\partial g_{ik}}{\partial u^j}u^{j\prime}u^{i\prime} + g_{mk}u^{m\prime\prime}. \end{aligned}$$

Therefore

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial u^k}u^{i\prime}u^{j\prime} - \frac{\partial}{\partial s}[g_{ik}u^{i\prime}] = 0$$

for $k = 1, 2$ implies

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial u^k}u^{i\prime}u^{j\prime} - \frac{\partial g_{ik}}{\partial u^j}u^{i\prime}u^{j\prime} - g_{mk}u^{m\prime\prime} = 0$$

for $k = 1, 2$, or using the notation of equations (35a, 35b, 35c)

$$\left\{\frac{1}{2}(\Gamma_{ikj} + \Gamma_{jki}) - (\Gamma_{kji} + \Gamma_{ijk})\right\} u^{i\prime}u^{j\prime} - g_{mk}u^{m\prime\prime} = 0 \qquad (*)$$

for $k = 1, 2$. Since $\Gamma_{ikj}u^{i\prime}u^{j\prime} = \Gamma_{jki}u^{i\prime}u^{j\prime}$ (interchanging dummy variables i and j) and $\Gamma_{kji} = \Gamma_{jki}$ (symmetry in the first and second coordinates) then

$$\left\{\frac{1}{2}(\Gamma_{ikj} + \Gamma_{jki}) - \Gamma_{kji}\right\} u^{i\prime}u^{j\prime} = \left\{\frac{1}{2}(\Gamma_{jki} + \Gamma_{jki}) - \Gamma_{jki}\right\} u^{i\prime}u^{j\prime} = 0$$

and the above equation $(*)$ becomes

$$\Gamma_{ijk}u^{i\prime}u^{j\prime} + g_{mk}u^{m\prime\prime} = 0$$

for $k = 1, 2$. Multiplying by g^{kr} and summing over k:

$$\Gamma_{ijk}g^{kr}u^{i\prime}u^{j\prime} + g^{kr}g_{mk}u^{m\prime\prime} = 0 \text{ or } \Gamma^r_{ij}u^{i\prime}u^{j\prime} + u^{r\prime\prime} = 0$$

for $r = 1, 2$. This is equation (32a) and therefore $\vec{\alpha}$ is a geodesic of M. ∎

Note. Again, Theorem I-9 along with Example 17 shows that the shortest distance between two points in the Euclidean plane is a "straight line." Theorem I-9 along with Example 18 show that the shortest distance between two points on a sphere is part of a great circle (explaining apparently unusual routes on international airline flights).

Note. The converse of Theorem I-9 is not true. That is, there may be a geodesic joining points which does not minimize distance. (Recall that we set $L'(0) \equiv 0$, but did not check $L''(0)$; we may have a maximum of L!) For example, we can travel the six miles from Johnson City to Jonesborough (along a very small piece of a geodesic), or we can travel in the opposite direction along a very large piece of a geodesic ($\approx 24{,}000$ miles) and travel around the world to get to Jonesborough (NOT a minimum distance).

Note. Not all surfaces may allow one to create a geodesic joining arbitrary points. For example, the Euclidean plane minus the origin does not admit a geodesic from $(1,1)$ to $(-1,-1)$.

Note. In the next theorem, we prove that for any point on a surface, there is a unique (directed) geodesic through that point in any direction.

Theorem I-10. Given a point $\vec{P}$ on a surface M and a unit tangent vector $\vec{v}$ at $\vec{P}$, there exists a unique geodesic $\vec{\alpha}$ such that $\vec{\alpha}(0) = \vec{P}$ and $\vec{\alpha}'(0) = \vec{v}$.

Proof. Let $\vec{P} = \vec{X}(u_0^1, u_0^2)$ and $\vec{v} = v^i \vec{X}_i(u_0^1, u_0^2)$. We need two functions $u^r(t)$, $r = 1, 2$ where

$$\begin{cases} u^{r\prime\prime} + \Gamma_{ij}^r u^{i\prime} u^{j\prime} = 0 & \text{for } r = 1, 2 \\ u^r(0) = u_0^r,\ u^{r\prime}(0) = v^r & \text{for } r = 1, 2. \end{cases}$$

This is a system of two ordinary differential equations in two unknown functions, each with two initial conditions. Such a system of IVPs has a unique solution (check out the chapter of an ODEs book entitled "Existence and Uniqueness Theorems") $u^r(t)$ for $r = 1, 2$. We now only need to establish that t represents arclength. With s equal to arclength,

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2 = g_{ij} u^{i\prime} u^{j\prime} \equiv f(t)$$

is the metric form and if we show this quantity is 1, then $|t| = s$ and t equals arclength (we need $\vec{\alpha}(0) = \vec{v}$ to eliminate the negative sign; this is insured by the initial conditions). Well,

$$f(0) \ = \ g_{ij}(u_0^1, u_0^2) u^{i\prime}(0) u^{j\prime}(0) = \vec{X}_i(u_0^1, u_0^2) \cdot \vec{X}_j(u_0^1, u_0^2) u^{i\prime}(0) u^{j\prime}(0)$$

$$
\begin{aligned}
&= \vec{X}_i(u_0^1,u_0^2)\cdot\vec{X}_j(u_0^1,u_0^2)v^iv^j = \left(v^i\vec{X}_i(u_0^1,u_0^2)\right)\cdot\left(v^j\vec{X}_j(u_0^1,u_0^2)\right)\\
&= \vec{v}\cdot\vec{v} = \|\vec{v}\|^2 = 1.
\end{aligned}
$$

Next,

$$f'(t) = \frac{\partial g_{ij}}{\partial u^k}u^{k\prime}u^{i\prime}u^{j\prime} + g_{ij}u^{i\prime\prime}u^{j\prime} + g_{ij}u^{i\prime}u^{j\prime\prime}.$$

Since

$$
\begin{aligned}
\frac{\partial g_{ij}}{\partial u^k} &= \Gamma_{ikj}+\Gamma_{jki} \text{ (equation (35b), page 60)}\\
&= \Gamma_{ik}^r g_{rj} + \Gamma_{jk}^r g_{ri} \text{ (equation (33), page 59)}\\
&= g_{jr}\Gamma_{ik}^r + g_{ir}\Gamma_{jk}^r \text{ (symmetry of } g_{ij})
\end{aligned}
$$

then

$$
\begin{aligned}
f'(t) &= (g_{jr}\Gamma_{ik}^r + g_{ir}\Gamma_{jk}^r)u^{i\prime}u^{j\prime}u^{k\prime} + g_{rj}u^{r\prime\prime}u^{j\prime} + g_{ir}u^{i\prime}u^{r\prime\prime}\\
&= g_{ir}u^{i\prime}(u^{r\prime\prime}+\Gamma_{jk}^r u^{j\prime}u^{k\prime}) + g_{rj}u^{j\prime}(u^{r\prime\prime}+\Gamma_{ik}^r u^{i\prime}u^{k\prime})\\
&= 0 \text{ (from the first condition of the ODE).}
\end{aligned}
$$

Therefore $f(t)$ is a constant and $f(t)=1$. Hence $\left(\frac{ds}{dt}\right)^2 = f(t) = 1$ and $t=s$ (that is, t is arclength). Therefore $\vec{\alpha}(s) = \vec{X}(u^1(s),u^2(s))$ is the desired geodesic.

∎

Example (Exercise 1.7.14(a)). If M has metric form $ds^2 = Edu^2 + Gdv^2$ with $E_u = G_u = 0$, then a geodesic on M satisfies

$$\frac{du}{dv} = \frac{h\sqrt{G}}{\sqrt{E}\sqrt{E-h^2}}$$

for some constant h (see Exercise 1.7.12). Use this above equation to show that a geodesic on the geographic sphere

$$\vec{X}(u,v) = (R\cos u\cos v, R\sin u\cos v, R\sin v)$$

satisfies

$$\frac{du}{dv} = \frac{h\sec^2 v}{\sqrt{R^2 - h^2\sec^2 v}} = \frac{h\sec^2 v}{\sqrt{R^2 - h^2 - h^2\tan^2 v}}$$

where h is a constant.

Solution. First,

$$\begin{aligned}
\vec{X}_1 &= (-R\sin u\cos v, R\cos u\cos v, 0) \\
\vec{X}_2 &= (-R\cos u\sin v, -R\sin u\sin v, R\cos v) \\
E &= g_{11} = \vec{X}_1 \cdot \vec{X}_1 = R^2\cos^2 v \\
G &= g_{22} = \vec{X}_2 \cdot \vec{X}_2 = R^2\sin^2 v + R^2\cos^2 v = R^2.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{du}{dv} &= \frac{h\sqrt{R^2}}{\sqrt{R^2\cos^2 v}\sqrt{R^2\cos^2 v - h^2}} \\
&= \frac{hR}{R\cos v\sqrt{R^2\cos^2 v - h^2}} \text{ since } v \in (-\pi/2, \pi/2) \\
&= \frac{h\sec v}{\cos v\sqrt{R^2 - h^2\sec^2 v}} = \frac{h\sec^2}{\sqrt{R^2 - h^2(1+\tan^2 v)}} \\
&= \frac{h\sec^2 v}{\sqrt{R^2 - h^2 - h^2\tan^2 v}}.
\end{aligned}$$

Example (Exercise 1.7.14(b)). Substitute $w = h\tan v$ and integrate the above equation to obtain $\cos(u - u_0) + \gamma\tan v = 0$ where u_0 and γ are constants.

Solution. With $w = h\tan v$, $dw = h\sec^2 v\, dv$ and so

$$u = \int \frac{h^2\sec^2 v}{\sqrt{R^2 - h^2 - h^2\tan^2 v}} dv$$

$$= -\int \frac{-1}{\sqrt{R^2 - h^2 - w^2}} dw = -\cos^{-1}\left(\frac{w}{\sqrt{R^2 - h^2}}\right) + u_0.$$

Therefore

$$\cos(u - u_0) = \frac{w}{\sqrt{R^2 - h^2}} = \frac{h \tan v}{\sqrt{R^2 - h^2}}.$$

With $\gamma = -h/\sqrt{R^2 - h^2}$ we have

$$\cos(u - u_0) + \gamma \tan v = 0.$$

Example (Exercise 1.7.14(c)). Show that the equation given in (b) when written in Cartesian coordinates is a linear equation of the form $\alpha x + \beta y + \gamma z = 0$ and so represents the intersection of the sphere with a plane passing through the origin (and therefore the geodesic is a great circle).

Solution. Multiplying by $R \cos v$ on each side of the equation gives $R \cos v \cos(u - u_0) + \gamma R \sin v = 0$ or $R \cos v(\cos u \cos u_0 + \sin u \sin u_0) + \gamma R \sin v = 0$ or

$$(\cos u_0) R \cos u \cos v + (\sin u_0) R \sin u \cos v + \gamma R \sin v = 0.$$

In cylindrical coordinates, ρ, θ, φ, we have the relationships $x = r \cos\theta \sin\varphi$, $y = \rho \sin\theta \sin\varphi$, and $z = \rho \cos\varphi$. Here, our R corresponds to cylindrical coordinates' ρ and our u corresponds to cylindrical coordinates' θ. However, our v does not correspond to φ of cylindrical coordinates but instead corresponds to $\pi/2 - \varphi$ (see my Calculus 3 notes `http://faculty.etsu.edu/gardnerr/2110/notes-12e/c15s7.pdf`). Our coordinates relate to rectangular coordinates as $x = R \cos u \sin(\pi/2 - v) = R \cos u \cos v$, $y = R \sin u \sin(\pi/2 - v) = R \sin u \cos v$, and $z = R \cos(\pi/2 - v) = R \sin v$. So the above equation is of the form $\alpha x + \beta y + \gamma z = 0$ where $\alpha = \cos u_0$, $\beta = \sin u_0$, and γ is a constant (as given in part (b)).

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