1.8 The Curvature Tensor and the *Theorema Egregium*

Recall. A property of a surface which depends only on the metric form is an intrinsic property. We have shown (Theorem I-5) that the Gauss curvature at a point $\vec{P}$ is $K(\vec{P}) = L/g$ where L is the Second Fundamental Form and g is the determinate of the matrix of the First Fundamental Form (or metric form). Therefore, to show that curvature is an intrinsic property of a surface, we need to show that L is a function of the g_{ij} (and their derivatives) which make up the metric form.

Recall. For a surface M determined by $\vec{X}(u^1, u^2)$ the coefficients of the Second Fundamental Form are

$$L_{ij} = \vec{X}_{ij} \cdot \vec{U} = \vec{X}_{ij} \cdot \frac{\vec{X}_1 \times \vec{X}_2}{\|\vec{X}_1 \times \vec{X}_2\|} \text{ (equation (20), page 44)}$$

and

$$L^i_j = L_{jk} g^{ki} \text{ (equation (27), page 54)}$$

and the Christoffel symbols are

$$\Gamma^r_{ij} = \frac{1}{2} g^{kr} \left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

(equation (37), Theorem 1.7.B). Also recall the formulas of Gauss

$$\vec{X}_{jk} = \Gamma^h_{jk} \vec{X}_h + L_{jk} \vec{U} \text{ (equation (17), page 43)}$$

and the formulas of Weingarten

$$\vec{U}_i = -L^j_i \vec{X}_j \text{ (equation (28), page 55).}$$

Lemma 1.8.A. The coefficients of the Second Fundamental Form and the Christoffel symbols are related as follows (for $h = 1, 2$):

$$\frac{\partial \Gamma^h_{ik}}{\partial u^j} - \frac{\partial \Gamma^h_{ij}}{\partial u^k} + \Gamma^r_{ik}\Gamma^h_{rj} - \Gamma^r_{ij}\Gamma^h_{rk} = L_{ik}L^h_j - L_{ij}L^h_k. \qquad (51)$$

Proof. Differentiating the formulas of Gauss:

$$\frac{\partial \vec{X}_{ik}}{\partial u^j} = \frac{\partial \Gamma^h_{ik}}{\partial u^j}\vec{X}_h + \Gamma^h_{ik}\frac{\partial \vec{X}_h}{\partial u^j} + \frac{\partial L_{ik}}{\partial u^j}\vec{U} + L_{ik}\frac{\partial \vec{U}}{\partial u^j}$$

or by defining $\partial/\partial u^j$ with a subscript of j

$$\vec{X}_{ikj} = \frac{\partial \Gamma^h_{ik}}{\partial u^j}\vec{X}_h + \Gamma^h_{ik}\vec{X}_{hj} + \frac{\partial L_{ik}}{\partial u^j}\vec{U} + L_{ik}\vec{U}_j.$$

Using the formulas of Gauss and Weingarten to rewrite $\vec{X}_{hj}$ and $\vec{U}_j$ we get

$$\vec{X}_{ikj} = \frac{\partial \Gamma^h_{ik}}{\partial u^j}\vec{X}_h + \Gamma^h_{ik}(\Gamma^r_{hj}\vec{X}_r + L_{hj}\vec{U}) + \frac{\partial L_{ik}}{\partial u^j}\vec{U} + L_{ik}(-L^h_j\vec{X}_h)$$

or (by interchanging h and r in the second term [since we are summing over both])

$$\begin{aligned}\vec{X}_{ikj} &= \frac{\partial \Gamma^h_{ik}}{\partial u^j}\vec{X}_h + \Gamma^r_{ik}(\Gamma^h_{rj}\vec{X}_h + L_{rj}\vec{U}) + \frac{\partial L_{ik}}{\partial u^j}\vec{U} + L_{ik}(-L^h_j\vec{X}_h) \\ &= \left(\frac{\partial \Gamma^h_{ik}}{\partial u^j} + \Gamma^r_{ik}\Gamma^h_{rj} - L_{ik}L^h_j\right)\vec{X}_h + \left(\Gamma^r_{ik}L_{rj} + \frac{\partial L_{ik}}{\partial u^j}\right)\vec{U}. \qquad (49)\end{aligned}$$

Interchanging j and k gives

$$\vec{X}_{ijk} = \left(\frac{\partial \Gamma^h_{ij}}{\partial u^k} + \Gamma^r_{ij}\Gamma^h_{rk} - L_{ij}L^h_k\right)\vec{X}_h + \left(\Gamma^r_{ij}L_{rk} + \frac{\partial L_{ij}}{\partial u^k}\right)\vec{U} \qquad (50)$$

(so we have $\vec{X}_{ijk}$ broken into a component normal to surface M and components which lie in the tangent plane to M at a given point; namely the components in directions $\vec{X}_1$ and $\vec{X}_2$). We have assumed that $\vec{X}$ is sufficiently continuous that

$\vec{X}_{ikj} = \vec{X}_{ijk}$ and so $\vec{X}_{ikj} - \vec{X}_{ijk} = \vec{0}$. Subtracting (50) from (49) and using the fact that the coefficients of $\vec{X}_1$ and $\vec{X}_2$ in the resultant are 0 we have

$$\frac{\partial \Gamma_{ik}^h}{\partial u^j} - \frac{\partial \Gamma_{ij}^h}{\partial u^k} + \Gamma_{ik}^r \Gamma_{rj}^h - \Gamma_{ij}^r \Gamma_{rk}^h - L_{ik} L_j^h + L_{ij} L_k^h = 0$$

for $h = 1, 2$ and the result follows. ∎

Definition. For a surface M with Christoffel symbols as above, define

$$R_{ijk}^h = \frac{\partial \Gamma_{ik}^h}{\partial u^j} - \frac{\partial \Gamma_{ij}^h}{\partial u^k} + \Gamma_{ik}^r \Gamma_{rj}^h - \Gamma_{ij}^r \Gamma_{rk}^h. \qquad (52)$$

These make up the *Riemann-Christoffel curvature tensor* (with $h = 1, 2$).

Note. Since the Christoffel symbols (Γ_{ij}^k's) are intrinsic properties of surface M by equation (37) of Theorem 1.7.B, the Riemann-Christoffel curvature tensor is also an intrinsic property of M.

Note. Interchanging j and k we trivially have $R_{ijk}^h = -R_{ikj}^h$. (53)

Theorem I-11. *Gauss' Theorema Egregium.*
The Gauss curvature of a surface is an intrinsic property. That is, the Gauss curvature of a surface is a function of the coefficients of the metric form and their derivatives.

Proof. From the Lemma 1.8.A and definition of R_{ijk}^h we have

$$R_{ijk}^h = L_{ik} L_j^h - L_{ij} L_k^h. \qquad (54)$$

Now define $R_{mijk} = g_{mh} R^h_{ijk} = g_{mr} R^r_{ijk}$. Then $R^r_{ijk} = g^{mr} R_{mijk}$. Now the Riemann-Christoffel curvature symbols R^h_{ijk} are intrinsic and therefore R_{mijk} are also intrinsic. Multiplying (54) by g_{mh} gives (summing over $h = 1, 2$)

$$g_{mh} R^h_{ijk} = g_{mh} L_{ik} L^h_j - g_{mh} L_{ij} L^h_k = g_{hm} L_{ik} L^h_j - g_{hm} L_{ij} L^h_k$$

or $R_{mijk} = L_{ik} L_{jm} - L_{ij} L_{km}$ since $g_{im} L^i_j = L_{jm}$ (equation (27′) in the notes and page 54 in the line after equation (27) in the book). In particular, with $m, j = 1$ and $i, k = 2$

$$\begin{aligned} R_{1212} &= L_{22} L_{11} - L_{21} L_{21} \\ &= L_{11} L_{22} - L_{12} L_{21} \text{ (since } L_{ij} = L_{ji}; \text{ by equation (20), page 44)} \\ &= \det(L_{ij}) = L. \end{aligned}$$

Therefore, since R_{mijk} are intrinsic, then L is intrinsic since, by Theorem I-5, $K = L/g = R_{1212}/g$ is intrinsic! ∎

Note. We now give an explicit equation for K in terms of the metric form.

Corollary 1.8.A. For a surface M determined by $\vec{X}(u, v) = \vec{X}(u^1, u^2)$ the curvature is given by

$$K = \frac{1}{g} \left[F_{uv} - \frac{1}{2} E_{vv} - \frac{1}{2} G_{uu} + (\Gamma^h_{12} \Gamma^r_{12} - \Gamma^h_{22} \Gamma^r_{11}) g_{rh} \right]$$

where

$$\begin{aligned} g_{11} &= \vec{X}_1 \cdot \vec{X}_1 = E \\ g_{12} &= \vec{X}_1 \cdot \vec{X}_2 = F = g_{21} \\ g_{22} &= \vec{X}_2 \cdot \vec{X}_2 = G \\ g &= \det(g_{ij}) \end{aligned}$$

and

$$\Gamma^r_{ij} = \frac{1}{2} g^{kr} \left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

where $(g_{ij})^{-1} = (g^{ij})$.

Proof. Since $R_{mijk} = g_{mh} R^h_{ijk}$ (by the definition of the Riemann-Christoffel curvature tensor; equation (55), page 76) and

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ik}}{\partial u^j} - \frac{\partial \Gamma^h_{ij}}{\partial u^k} + \Gamma^r_{ik}\Gamma^h_{rj} - \Gamma^r_{ij}\Gamma^h_{rk}$$

(equation (52), page 75) then

$$g_{mh} R^h_{ijk} = g_{mh} \frac{\partial \Gamma^h_{ik}}{\partial u^j} + g_{mh}\Gamma^r_{ik}\Gamma^h_{rj} - g_{mh} \frac{\partial \Gamma^h_{ij}}{\partial u^k} - g_{mh}\Gamma^r_{ij}\Gamma^h_{rk}$$

or

$$\begin{aligned} R_{mijk} &= g_{mh} \frac{\partial \Gamma^h_{ik}}{\partial u^j} + g_{hm}\Gamma^h_{rj}\Gamma^r_{ik} - g_{mh} \frac{\partial \Gamma^h_{ij}}{\partial u^k} - g_{hm}\Gamma^h_{rk}\Gamma^r_{ij} \text{ (since } g_{hm} = g_{mh}) \\ &= g_{mh} \frac{\partial \Gamma^h_{ik}}{\partial u^j} + \Gamma_{rjm}\Gamma^r_{ik} - g_{mh} \frac{\partial \Gamma^h_{ij}}{\partial u^k} - \Gamma_{rkm}\Gamma^r_{ij} \qquad (*) \end{aligned}$$

since $\Gamma_{ijk} = \Gamma^r_{ij} g_{rk}$ (equation (33), page 59). Now, interchanging the indices in equation (33) we have $g_{mh}\Gamma^h_{ik} = \Gamma_{ikm}$ or differentiating with respect to u^j

$$\frac{\partial g_{mh}}{\partial u^j}\Gamma^h_{ik} + g_{mh}\frac{\partial \Gamma^h_{ik}}{\partial u^j} = \frac{\partial \Gamma_{ikm}}{\partial u^j}$$

or

$$g_{mh}\frac{\partial \Gamma^h_{ik}}{\partial u^j} = \frac{\partial \Gamma_{ikm}}{\partial u^j} - \Gamma^h_{ik}\frac{\partial g_{hm}}{\partial u^j}. \qquad (**)$$

Now using $(**)$ in $(*)$ we have

$$R_{mijk} = \left(\frac{\partial \Gamma_{ikm}}{\partial u^j} - \Gamma^h_{ik}\frac{\partial g_{hm}}{\partial u^j} \right) + \Gamma_{rjm}\Gamma^r_{ik} - \left(\frac{\partial \Gamma_{ijm}}{\partial u^k} - \Gamma^h_{ij}\frac{\partial g_{hm}}{\partial u^k} \right) - \Gamma_{rkm}\Gamma^r_{ij}.$$

Replacing r by h in the products of Γ's gives

$$R_{mijk} = \left(\frac{\partial \Gamma_{ikm}}{\partial u^j} - \Gamma^h_{ik}\frac{\partial g_{hm}}{\partial u^j}\right) + \Gamma_{hjm}\Gamma^h_{ik} - \left(\frac{\partial \Gamma_{ijm}}{\partial u^k} - \Gamma^h_{ij}\frac{\partial g_{hm}}{\partial u^k}\right) - \Gamma_{hkm}\Gamma^h_{ij}.$$

Now

$$\Gamma_{ikm} = \frac{1}{2}\left(\frac{\partial g_{im}}{\partial u^k} + \frac{\partial g_{mk}}{\partial u^i} - \frac{\partial g_{ki}}{\partial u^m}\right) \text{ (equation (36), page 60)}$$

and

$$\begin{aligned}
\frac{\partial g_{hm}}{\partial u^j} &= \Gamma_{hjm} + \Gamma_{mjh} \text{ (equation (35), page 60)}\\
&= \Gamma_{hjm} + \Gamma^r_{mj} g_{rh} \text{ (equation (33), page 59)}
\end{aligned}$$

so

$$\begin{aligned}
R_{mijk} &= \frac{\partial}{\partial u^j}\left[\frac{1}{2}\left(\frac{\partial g_{im}}{\partial u^k} + \frac{\partial g_{mk}}{\partial u^i} - \frac{\partial g_{ki}}{\partial u^m}\right)\right] + \Gamma^h_{ik}\Gamma_{hjm} - \Gamma^h_{ik}(\Gamma_{hjm} + \Gamma^r_{mj}g_{rh})\\
&\quad -\frac{\partial}{\partial u^k}\left[\frac{1}{2}\left(\frac{\partial g_{im}}{\partial u^j} + \frac{\partial g_{mj}}{\partial u^i} - \frac{\partial g_{ji}}{\partial u^m}\right)\right] - \Gamma^h_{ij}\Gamma_{hkm} + \Gamma^h_{ij}(\Gamma_{hkm} + \Gamma^r_{mk}g_{rh})\\
&= \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial u^j\,\partial u^k} + \frac{\partial^2 g_{mk}}{\partial u^j\,\partial u^i} - \frac{\partial^2 g_{ki}}{\partial u^j\,\partial u^m}\right) + \Gamma^h_{ik}\Gamma_{hjm} - \Gamma^h_{ik}(\Gamma_{hjm} + \Gamma^r_{mj}g_{rh})\\
&\quad -\frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial u^k\,\partial u^j} + \frac{\partial^2 g_{mj}}{\partial u^k\,\partial u^i} - \frac{\partial^2 g_{ji}}{\partial u^k\,\partial u^m}\right) - \Gamma^h_{ij}\Gamma_{hkm} + \Gamma^h_{ij}(\Gamma_{hkm} + \Gamma^r_{mk}g_{rh})\\
&= \frac{1}{2}\left(\frac{\partial^2 g_{km}}{\partial u^j\,\partial u^i} - \frac{\partial^2 g_{jm}}{\partial u^i\,\partial u^k} + \frac{\partial^2 g_{ij}}{\partial u^k\,\partial u^m} - \frac{\partial^2 g_{ik}}{\partial u^j\,\partial u^m}\right) + (\Gamma^h_{ij}\Gamma^r_{mk} - \Gamma^h_{ik}\Gamma^r_{mj})g_{rh}.
\end{aligned}$$

So with $m = j = 1$ and $i = k = 2$

$$\begin{aligned}
R_{1212} &= \frac{1}{2}\left(\frac{\partial^2 g_{21}}{\partial u^1\,\partial u^2} - \frac{\partial^2 g_{11}}{\partial u^2\,\partial u^2} + \frac{\partial^2 g_{21}}{\partial u^2\,\partial u^1} - \frac{\partial^2 g_{22}}{\partial u^1\,\partial u^1}\right)\\
&\quad +(\Gamma^h_{21}\Gamma^r_{12} - \Gamma^h_{22}\Gamma^r_{11})g_{rh}\\
&= \frac{1}{2}(F_{uv} - E_{vv} + F_{uv} - G_{uu}) + (\Gamma^h_{21}\Gamma^r_{12} - \Gamma^h_{22}\Gamma^r_{11})g_{rh}.
\end{aligned}$$

Since $K = R_{1212}/g$ (equation (57), page 76), and the result follows. ∎

Corollary 1.8.B. For a surface M determined by $\vec{X}(u,v)$ with orthogonal coordinates ($\vec{X}_1 \cdot \vec{X}_2 = F = 0$) the curvature is

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\left[\frac{G_u}{\sqrt{EG}}\right] + \frac{\partial}{\partial v}\left[\frac{E_v}{\sqrt{EG}}\right]\right).$$

Proof. With $F = 0$ and equation (40) of page 62 (which gives the Christoffel symbols in an orthogonal coordinate system in terms of E and G) we have

$$\begin{aligned}
K &= \frac{1}{EG}\left\{-\frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} + ((\Gamma^1_{12})^2 - \Gamma^1_{22}\Gamma^1_{11})g_{11} + ((\Gamma^2_{12})^2 - \Gamma^2_{22}\Gamma^2_{11})g_{22}\right\} \\
&\quad \text{(since } g_{12} = g_{21} = 0 \text{ and } \det(g_{ij}) = g_{11}g_{22} = EG\text{)} \\
&= \frac{1}{EG}\left\{-\frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} + \left(\left(\frac{E_v}{2E}\right)^2 - \left(-\frac{G_u}{2E}\right)\left(\frac{E_u}{2E}\right)\right)E\right. \\
&\quad \left. + \left(\left(\frac{G_u}{2G}\right)^2 - \left(\frac{G_v}{2G}\right)\left(-\frac{E_v}{2G}\right)\right)G\right\} \\
&= \frac{1}{EG}\left\{-\frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} + \left(\frac{E_v^2}{4E^2} + \frac{E_uG_u}{4E^2}\right)E + \left(\frac{G_u^2}{4G^2} + \frac{E_vG_v}{4G^2}\right)G\right\} \\
&= -\frac{1}{EG}\left\{\frac{1}{2}E_{vv} + \frac{1}{2}G_{uu} - \frac{EE_v^2 + EE_uG_u}{4E^2} - \frac{GG_u^2 + E_vGG_v}{4G^2}\right\} \\
&= \frac{-1}{2EG\sqrt{EG}}\left\{\sqrt{EG}E_{vv} + \sqrt{EG}G_{uu} - \sqrt{EG}\left(\frac{E_v^2 + E_uG_u}{2E}\right)\right. \\
&\quad \left. - \sqrt{EG}\left(\frac{G_u^2 + E_vG_v}{2G}\right)\right\} \\
&= \frac{-1}{2EG\sqrt{EG}}\left\{\sqrt{EG}E_{vv} + \sqrt{EG}G_{uu} - \frac{GE_v^2 + E_uGG_u}{2\sqrt{EG}}\right. \\
&\quad \left. - \frac{EG_u^2 + EE_vG_v}{2\sqrt{EG}}\right\} \\
&= \frac{-1}{2EG\sqrt{EG}}\left\{\sqrt{EG}E_{vv} + \sqrt{EG}G_{uu} - \frac{G_u(EG_u + E_uG)}{2\sqrt{EG}}\right. \\
&\quad \left. - \frac{E_v(EG_v + E_vG)}{2\sqrt{EG}}\right\}
\end{aligned}$$

$$= \frac{-1}{2\sqrt{EG}} \left\{ \frac{\sqrt{EG}G_{uu} - \frac{G_u(EG_u+E_uG)}{2\sqrt{EG}}}{EG} + \frac{\sqrt{EG}E_{vv} - \frac{E_v(EG_v+E_vG)}{2\sqrt{EG}}}{EG} \right\}$$

$$= \frac{-1}{2\sqrt{EG}} \left\{ \frac{\partial}{\partial u}\left[\frac{G_u}{\sqrt{EG}}\right] + \frac{\partial}{\partial v}\left[\frac{E_v}{\sqrt{EG}}\right]\right\}.$$

∎

Note. The equation given in the previous corollary will be useful in the exercises in this section.

Note. Some symmetry relations in R_{mijk} are given at the end of the section.

Example (Exercise 2, page 80). Let

$$\vec{X}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

be a surface of revolution whose profile curve $\vec{\alpha}(u) = (f(u), 0, g(u))$ has unit speed. Show that $K = -f''/f$.

Solution. By Exercise 1.4.5, page 39, $E = g_{11} = (f'(u))^2 + (g'(u))^2$, $F = g_{12} = g_{21} = 0$ (coordinates are orthogonal), $G = g_{22} = (f(u))^2$. So by equation (59), page 78,

$$K = \frac{-1}{2\sqrt{(f(u))^2\{(f'(u))^2+(g'(u))^2\}}}$$
$$\times \left\{ \frac{\partial}{\partial u}\left[\frac{2f(u)f'(u)}{\sqrt{(f(u))^2\{(f'(u))^2+(g'(u))^2\}}}\right] + \frac{\partial}{\partial v}[0]\right\}.$$

Now assuming $\|\vec{\alpha}'\| = \sqrt{(f'(u))^2 + (g'(u))^2} = 1$ and $f(u) \geq 0$:

$$K = \frac{-1}{2f(u)} \left\{ \frac{\partial}{\partial u}[2f'(u)] \right\} = \frac{-f''(u)}{f(u)}.$$

Example (Exercise 5 (b), page 81). The *pseudosphere* may be represented as the surface of revolution

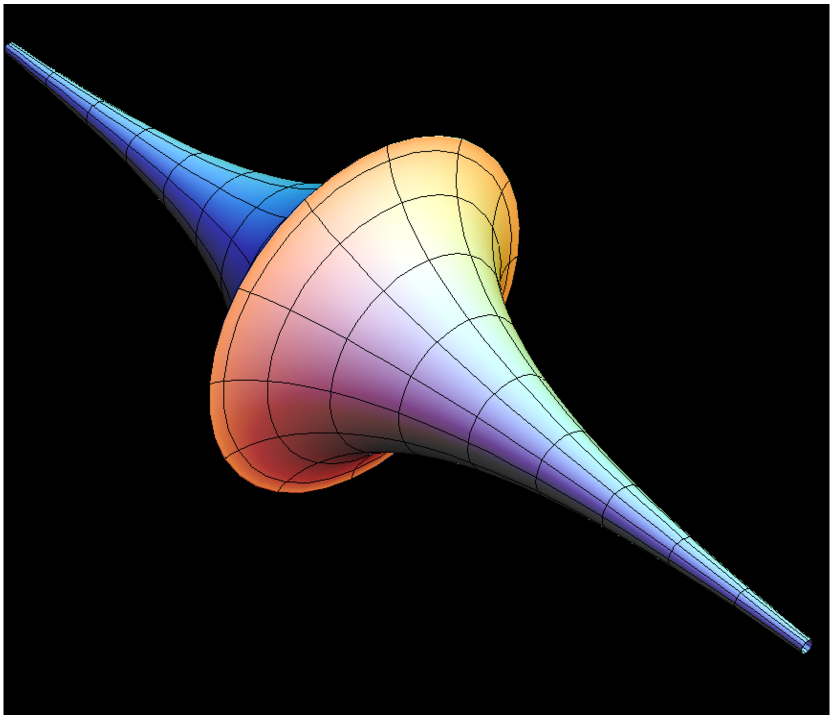
$$\vec{X}(u,v) = \left(a \sin u \cos v, a \sin u \sin v, a \left[\cos u + \ln \left(\tan \frac{u}{2} \right) \right] \right)$$

for $u \in (0, \pi/2)$. Show that $K = -1/a^2$ (and so the pseudosphere has constant negative curvature).

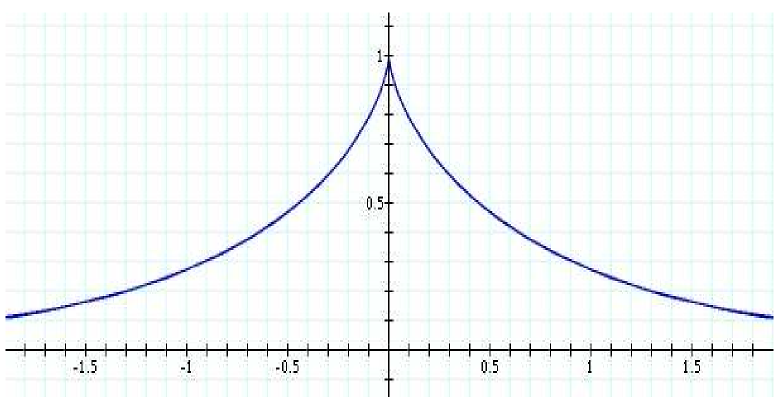
Solution. In Exercise 5 (a), you will show that $E = a^2 \cot^2 u$ and $G = a^2 \sin^2 u$. Therefore by equation (59), page 78:

$$\begin{aligned}
K &= \frac{-1}{2\sqrt{a^4 \cot^2 u \sin^2 u}} \left(\frac{\partial}{\partial u} \left[\frac{2a^2 \sin u \cos u}{\sqrt{a^4 \cot^2 u \sin^2 u}} \right] + \frac{\partial}{\partial v}[0] \right) \\
&= \frac{-1}{2a^4 \cot^2 u \sin^2 u} \frac{\partial}{\partial u} \left[\frac{2a^2 \sin u \cos u}{a^4 \cot^2 u \sin^2 u} \right] \text{ since } u \in (0, \pi/2) \\
&= \frac{-1}{2a^4 \cot^2 u \sin^2 u} \frac{\partial}{\partial u} [2 \sin u] \\
&= \frac{-1}{2a^4 \cot^2 u \sin^2 u} (2 \cos u) = \frac{-\cot u}{a^2 \cot u} = \frac{-1}{a^2}.
\end{aligned}$$

Note. The pseudosphere looks like (from: `http://virtualmathmuseum.org/Surface/pseudosphere/pseudosphere.html`):



Note. In the $zx-$plane, the profile curve of the pseudosphere is (from: `http://xahlee.info/SpecialPlaneCurves_dir/Tractrix_dir/tractrix.html`):

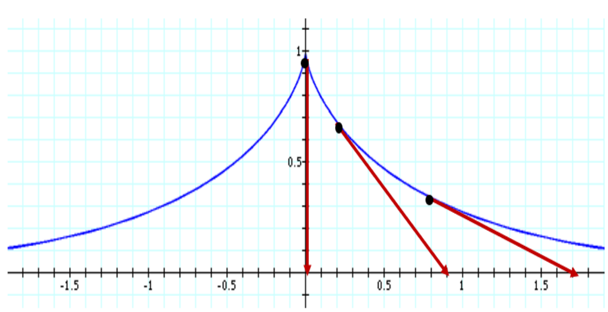


We get the point $(z, x) = (0, 1)$ for $u = \pi/2$. Let's calculate the arclength s for u

ranging from $\pi/2$ to u^*:

$$\begin{aligned}
s &= -\int_{\pi/2}^{u^*} \sqrt{(x'(u))^2 + (z'(u))^2}\, du \text{ (since } u^* < \pi/2)\\
&= \int_{u^*}^{\pi/2} \sqrt{\cos^2(u) + \left(-\sin u + \frac{\sec^2(u/2)}{2\tan(u/2)}\right)^2}\, du\\
&= \int_{u^*}^{\pi/2} \sqrt{\cos^2 u + \left(-\sin u + \frac{1}{2\sin(u/2)\cos(u/2)}\right)^2}\, du\\
&= \int_{u^*}^{\pi/2} \sqrt{\cos^2 u + \left(-\sin u + \frac{1}{\sin u}\right)^2}\, du\\
&= \int_{u^*}^{\pi/2} \sqrt{\cos^2 u + \sin^2 u - 2 + \csc^2 u}\, du\\
&= \int_{u^*}^{\pi/2} \sqrt{\csc^2 u - 1}\, du = \int_{u^*}^{\pi/2} |\cot u|\, du\\
&= \int_{u^*}^{\pi/2} \cot u\, du = \ln(\sin u)\Big|_{u^*}^{\pi/2} = -\ln(\sin u^*).
\end{aligned}$$

Therefore $\exp(-\text{arclength}) = e^{-s} = e^{-(-\ln(\sin u^*))} = \sin u^* = x^*$. So we have $x = e^{-s}$ where s is arclength. This curve is called a *tractrix*. It can be generated by placing a box at point $(0, 1)$ and dragging it by attaching a 1 unit rope and pulling along the $z-$axis (therefore the tangent line at any point meets the $z-$axis 1 unit from the point of tangency; based on the previous figure):



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