# 3.2 Gravity as Spacetime Curvature

Note. Thus far we have only considered uniform gravitational fields. In nature, gravitational "force" varies from point to point and so the acceleration due to this force is not uniform. First consider two particles "side by side" in a gravitational field (see below) directed towards a point (or the center of a sphere). As the particles fall, they will be drawn closer together. Second, consider two particles in a gravitational field which are separated vertically. This time, the difference in the forces produces a growing separation between the particles as they fall. In both cases, the behavior is an example of *tidal effects.* Therefore, a freely falling "space capsule" does not behave exactly like an inertial frame. However, over short spans of time it is a good approximation of an inertial frame.
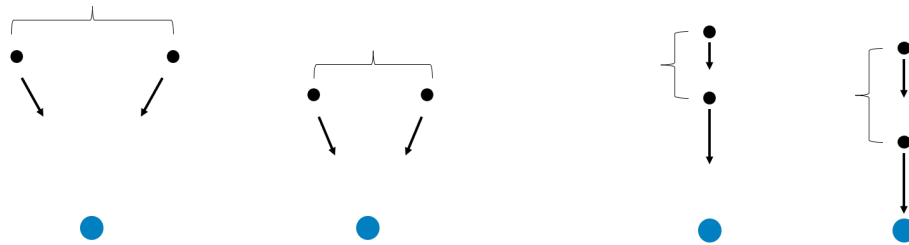
Note. We rephrase the Principle of Equivalence as:

> For each spacetime point (i.e. event), and for a given degree
> of accuracy, there exists a frame of reference in which in a
> certain region of space and for a certain interval of time,
> the effects of gravity are negligible and the frame is inertial
> to the degree of accuracy specified.

Such a frame is called a *locally inertial frame* (at the given event) and an observer in such a frame is a *locally inertial observer.*

Note. The convergence and divergence of particles as described above has a geometric analog. On a positively curved surface, "parallel" geodesics converge, and on a negatively curved surface, "parallel" geodesics diverge (see page 175).

Note. Einstein proposed that gravity is not a force, but a curvature of spacetime! He hypothesized that free particles (and photons) follow geodesics in a curved spacetime.

Definition. Define the *matrix of the Lorentz metric* as

$$(\eta_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note. It is also common to take the values of η_{ij} to be the negatives of what are given here. For example, in his 1916 paper Einstein himself uses this convention; see his Equation (4) on page 120 of the translation of his paper in *The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity* by H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, Dover Publications (1952).

Note. A locally inertial observer in a locally inertial frame can set up a system of coordinates where the interval satisfies

$$d\tau^2 \approx \eta_{ij} du^i du^j = (du^0)^2 - (du^1)^2 - (du^2)^2 - (du^3)^2.$$

More precisely, a coordinate system (u^i) can be set up at a point $\vec{P}$ in a locally inertial frame such that

$$d\tau^2 = g_{ij} du^i du^j$$

and the functions g_{ij} satisfy

$$
\begin{aligned}
g_{ij}(\vec{P}) &= \eta_{ij} \\
\left. \frac{\partial g_{ij}}{\partial u^k} \right|_{\vec{P}} &= 0.
\end{aligned}
$$

These two conditions imply that the metric is the Lorentz metric at $\vec{P}$, and that the metric differs little from the Lorentz metric near $\vec{P}$ (i.e. the rate of change of the g_{ij}'s is small near $\vec{P}$).

Definition. A coordinate system such that $d\tau^2 = g_{ij} du^i du^j$ where g_{ij} are functions of u^i $(i = 0, 1, 2, 3)$ satisfying

$$
\begin{aligned}
g_{ij}(\vec{P}) &= \eta_{ij} \\
\left. \frac{\partial g_{ij}}{\partial u^k} \right|_{\vec{P}} &= 0
\end{aligned}
$$

for $i, j, k = 0, 1, 2, 3$, at some point $\vec{P}$ is a *locally Lorentz coordinate system* at $\vec{P}$.

Note. We are again back to the idea of a manifold. The Principle of Equivalence tells us that in the 4-manifold of general relativity, small local neighborhoods look like the "flat" 4-manifold of special relativity. The departure (on a larger scale) of $d\tau^2$ from the Lorentz metric is due to the nonuniformity of gravity and is the result (as we will see) of the curvature of spacetime!

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