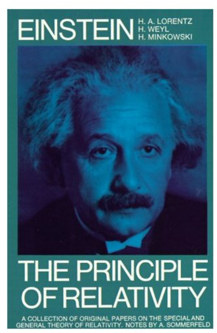
3.7 The Field Equations

Note. We now want a set of equations relating the metric coefficients $g_{\mu\nu}$ which determine the curvature of spacetime due to the distribution of matter in spacetime. Einstein accomplished this in his "Die Grundlage der allgemeinen Relativitätstheorie" ("The Foundation of the General Theory of Relativity") in *Annalen der Physik* (*Annals of Physics*) **49**(7), 1916. As commented earlier, this is reprinted in *The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity* by H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, Dover Publications (1952), pages 109–164. We frequently quote this reference in this section (calling it simply "the Dover book").



Note. Consider a mass M at the origin of a 3-dimensional system. Let $\vec{X} = (x, y, z) = (x(t), y(t), z(t))$, and $\|\vec{X}\| = \sqrt{x^2 + y^2 + z^2} = r$. Let $\vec{u}_r$ be the unit radial vector $\vec{X}/r$. Under Newton's laws, the force $\vec{F}$ on a particle of mass m located at $\vec{X}$ is

$$\vec{F} = -\frac{Mm}{r^2}\vec{u}_r = m\frac{d^2\vec{X}}{dt^2}.$$

Therefore $\dfrac{d^2\vec{X}}{dt^2} = -\dfrac{M}{r^2}\vec{u}_r$.

Definition. For a particle at point (x, y, z) in a coordinate system with mass M at the origin, define the *potential function* $\Phi = \Phi(r)$ as

$$\Phi(r) = -\frac{M}{r}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

Theorem. The potential function satisfies Laplace's equation

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

at all points except the origin.

Proof. First

$$\frac{\partial r}{\partial x^i} = \frac{\partial}{\partial x^i}[(\vec{X} \cdot \vec{X})^{1/2}] = \frac{2x^i}{2(\vec{X} \cdot \vec{X})^{1/2}} = \frac{x^i}{r}$$

and

$$\frac{\partial \Phi}{\partial x^i} = \frac{\partial \Phi}{\partial r} \frac{\partial r}{\partial x^i}.$$

Therefore

$$\begin{aligned} -\nabla \Phi &= -\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right) \\ &= -\frac{M}{r^2} \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) = -\frac{M}{r^2} \vec{u}_r = \frac{d^2 \vec{X}}{dt^2}. \end{aligned}$$

Comparing components,

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i}. \qquad (122)$$

Differentiating the relationship

$$\frac{\partial \Phi}{\partial x^i} = \frac{\partial}{\partial x^i} \left[-\frac{M}{r} \right]$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x^i}\left[\frac{-M}{((x^1)^2+(x^2)^2+(x^3)^2)^{1/2}}\right] \\
&= \frac{-1(-1/2)M(2x^i)}{((x^1)^2+(x^2)^2+(x^3)^2)^{3/2}} = \frac{Mx^i}{r^3}
\end{aligned}
$$

gives

$$
\begin{aligned}
\frac{\partial^2\Phi}{(\partial x^i)^2} &= M\left(\frac{r^3 - x^i[(3/2)r(2x^i)]}{r^6}\right) \\
&= M\frac{r^3 - 3r(x^i)^2}{r^6} = \frac{M}{r^5}(r^2 - 3(x^i)^2).
\end{aligned}
$$

Summing over $i = 1, 2, 3$ gives

$$
\nabla^2\Phi = \frac{M}{r^5}\{(r^2 - 3(x^1)^2) + (r^2 - 3(x^2)^2) + (r^2 - 3(x^3)^2)\} = 0.
$$

∎

Note. In the case of a finite number of point masses, the Laplace's equation still holds, only Φ is now a sum of terms (one for each particle).

Note. In general relativity, we replace equation (122) with

$$
\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad (125)
$$

where the Christoffel symbols are

$$
\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right).
$$

Note. Comparing equations (122) and (125), we see that

$$\frac{\partial \Phi}{\partial x^i} \quad \text{and} \quad \Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}$$

play similar roles. As the text says, "...in a sense then, the metric coefficients play the role of gravitational potential functions in Einstein's theory." This is explained by Einstein in his paper as (see page 143 of the Dover book):

> "...the equation of motion of the point with respect to [any chosen system of coordinates] K_1, becomes
>
> $$\frac{d^2 x_r}{ds^2} = \Gamma^{r}_{\mu\nu}\frac{dx_{\mu}}{ds}\frac{dx_{\nu}}{ds}. \qquad (46)$$
>
> We now make the assumption, which readily suggests itself, that this covariant system of equations also defines the motion of the point in the gravitational field in the case when there is no system of reference K_0, with respect to which the special theory of relativity holds good in a finite region. We have all the more justification for this assumption as (46) contains only *first* derivatives of the $g_{\mu\nu}$, between which even in the special case of the existence of K_0, no relations subsist.
>
> If the $\Gamma^{r}_{\mu\nu}$ vanish, then the point moves uniformly in a straight line. These quantities therefore condition the deviation of the motion from uniformity. They are the components of the gravitational field."

Note. Trying to come up with a result analogous to Laplace's equation and treating the $g_{\mu\nu}$'s as a potential function, we might desire a field equation of the form $G = 0$ where G involves the second partials of the $g_{\mu\nu}$'s.

Note. "It turns out" that the only tensors that are constructible from the metric coefficients $g_{\mu\nu}$ and their first and second derivatives are those that are functions of $g_{\mu\nu}$ and the components of $R^\lambda_{\mu\nu\sigma}$ of the curvature tensor.

Note. We want the field equations to have the flat spacetime of special relativity as a special case. In this special case, the $g_{\mu\nu}$ are constants and so we desire $R^\lambda_{\mu\nu\sigma} = 0$ for each index ranging from 0 to 3 (since the partial derivatives of the $g_{\mu\nu}$ are involved). However, "it can be shown" that this system of PDEs (in the unknown $g_{\mu\nu}$'s) implies that the $g_{\mu\nu}$'s are constant (and therefore that we are under the flat spacetime of special relativity...we could use some details to verify this!).

Definition. The *Ricci tensor* is obtained from the curvature tensor by summing over one index in the Riemann-Christoffel curvature tensor:

$$R_{\mu\nu} = R^\lambda_{\mu\nu\lambda} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\Gamma^\lambda_{\beta\lambda}.$$

Note. In his 1916 paper, Einstein does not use the Ricci tensor by name, but does refer the Riemann-Christoffel tensor (in fact, the title of Section 12 of his paper is "The Riemann-Christoffel Tensor"). We have denoted the Riemann-Christoffel tensor as R^h_{ijk} in our section 1.8, but Einstein denotes it as $B^\rho_{\mu\sigma\tau}$ (see pages 141 and 142 of the Dover book). He then states (in his Equation (44)) that

$$G_{\mu\nu} = B^\rho_{\mu\nu\rho} = R_{\mu\nu} + S_{\mu\nu}$$

where (notice that Einstein's Equation (45) gives $\Gamma^\tau_{\mu\nu} = -\{\mu\nu, \tau\}$)

$$R_{\mu\nu} = -\frac{\partial}{\partial x_\alpha}\{\mu\nu, \alpha\} + \{\mu\alpha, \beta\}\,\{\nu\beta, \alpha\} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\alpha} + \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta}$$

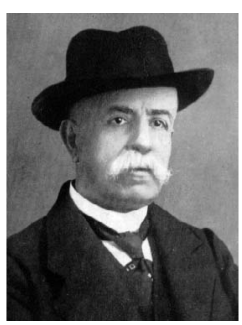
and

$$S_{\mu\nu} = \frac{\partial^2 \log\sqrt{-g}}{\partial x_\mu\,\partial x_\nu} - \{\mu\nu,\alpha\}\frac{\partial^2 \log\sqrt{-g}}{\partial x_\mu\,\partial x_\nu}$$

$$= \frac{\partial^2 \log\sqrt{-g}}{\partial x_\mu\,\partial x_\nu} + \Gamma^\alpha_{\mu\nu}\frac{\partial^2 \log\sqrt{-g}}{\partial x_\mu\,\partial x_\nu} = -\frac{\partial^2 \log\sqrt{g}}{\partial x_\mu\,\partial x_\nu} - \Gamma^\alpha_{\mu\nu}\frac{\partial^2 \log\sqrt{g}}{\partial x_\mu\,\partial x_\nu}\ \text{(HMMM)}.$$

Einstein then argues that coordinates can be chosen so that $\sqrt{-g} = 1$, in which case $S_{\mu\nu} = 0$ and then $G_{\mu\nu} = R_{\mu\nu}$, so that $G_{\mu\nu}$ is then the Ricci tensor. Einstein states after this that: "On this account I shall hereafter give all relations in the simplified form which this specialization of the choice of co-ordinates brings with it. It will then be an easy matter to revert to the *generally* covariant equations, if this seems desirable in a special case." (See page 142 of the Dover book.) Einstein's $R_{\mu\nu}$ does not agree with our $R_{\mu\nu}$. In fact, Einstein's $G_{\mu\nu}$ is our $R_{\mu\nu}$. We will further elaborate on this at the end of this section.

Note. The Ricci tensor (pronounced "REE-CHEE") is named for Gregorio Ricci-Curbastro (1853–1925). He developed "absolute differential calculus" between 1884 and 1894. This is work based on initial work of Gauss, Riemann (in particular, Riemann's 1854 "On the Hypotheses at the Foundations of Geometry"), and Elwin Christoffel (in particular, an 1868 paper he published in *Crelle's Journal*). He presented his work in four publications between 1888 and 1892. This became the foundation of the tensor analysis used by Einstein in his general relativity work. See the MacTutor History of Math archive biography for Ricci-Curbastro.



Note. Einstein chose as his field equations the system of second order PDEs $R_{\mu\nu} = 0$ for $\mu, \nu = 0, 1, 2, 3$. More explicitly:

Definition. Einstein's *field equations* for general relativity are the system of second order PDEs

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\beta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\lambda}_{\beta\lambda} = 0$$

where

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x^{\nu}} + \frac{\partial g_{\nu\beta}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right).$$

Therefore, the field equations are a system of second order PDEs in the unknown function $g_{\mu\nu}$ (16 equations in 16 unknown functions). The $g_{\mu\nu}$ determine the metric form of spacetime and therefore all intrinsic properties of the 4-dimensional semi-Riemannian manifold that is spacetime (such as curvature)!

Note. Faber's presentation seems a bit of a simplification of what appears in Einstein's original 1916 paper. For the historical importance of the 1916 work (and in commemoration of the centennial of this work, since these notes are being written in 2016), we include Einstein's statement of his "general field equations" from pages 148 and 149 of the Dover book, which corresponds to Section 16 of the paper, "The General Form of the Field Equations of Gravitation":

> "For if we consider a complete system (e.g. the solar system), the total mass of the system, and therefore its total gravitating action as well, will depend on the total energy of the system, and therefore on the ponderable energy together with the gravitational energy. This will allow itself to be expressed by introducing [...], in place of the energy-components of the gravitational field alone, the sums $t^{\sigma}_{\mu} + T^{\sigma}_{\mu}$ of the energy-components of matter and of gravitational field. Thus [...] we

obtain the tensor equation

$$\left.\begin{array}{rcl} \frac{\partial}{\partial x_\alpha}(g^{\sigma\beta}T^\alpha_{\mu\beta}) & = & -\kappa[(t^\sigma_\mu + T^\sigma_\mu) - \frac{1}{2}\delta^\sigma_\mu(t+T)], \\ \sqrt{-g} & = & 1 \end{array}\right\} \qquad (52)$$

where we have set $T = T^\mu_\mu$ (Laue's scalar). [Parameter g is the determinant of matrix $[g_{\mu\nu}]$.] These are the required general field equations of gravitation in mixed form. Working back from these, we have [. . .]

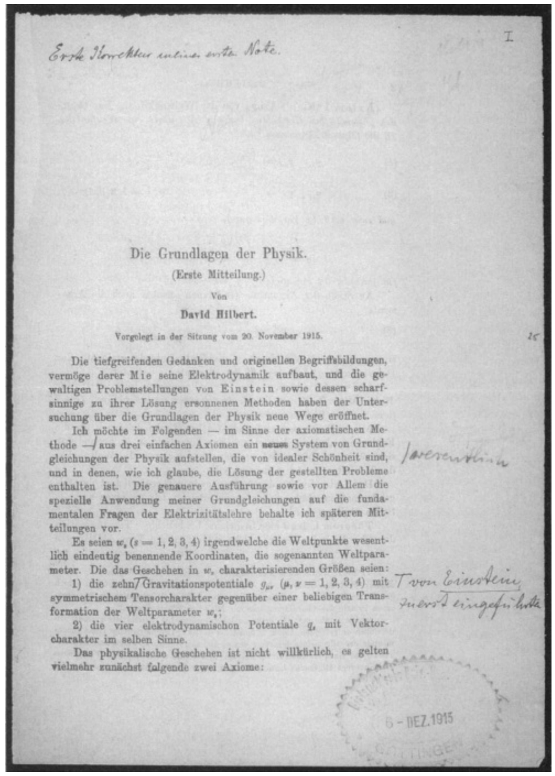
$$\left.\begin{array}{rcl} \frac{\partial}{\partial x_\alpha}\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} & = & -\kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T), \\ \sqrt{-g} & = & 1 \end{array}\right\}. \qquad (53)$$

[In the first part of (53), κ is a constant, $T_{\mu\nu}$ is the stress-energy tensor of matter and T is the trace of $T_{\mu\nu}$. Notice that the left-hand side is just the Ricci tensor, so that this reduces to another form which you might see in the literature: $R_{\mu\nu} = -\kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$.]

It must be admitted that this introduction of the energy tensor of matter is not justified by the relativity postulate alone. For this reason we have here deduced it from the requirement that the energy of the gravitational field shall act gravitatively in the same way as any other kind of energy. But the strongest reason for the choice of these equations lies in their consequence, that the equations of conservation of momentum and energy [. . .] hold good for the components of the total energy."

Note. Albert Einstein worked for eight years (1907 to 1915) on his general theory of relativity. Einstein visited Göttingen, Germany (he was living in Berlin at the time) in June and early July of 1915 and gave six lectures on his theory as it stood then. David Hilbert (1862–1943) was in attendance and he began exploring an approach to a new theory of gravity based on Gustav Mie's electromagnetic theory of matter and Einstein's work. The early history of general relativity involves an interesting interaction between Einstein and Hilbert and a potential claim of priority on the field equations by Hilbert! These historical notes are based on "Belated Decision in the Hilbert-Einstein Priority Dispute" by Leo Corry, Jürgen Renn and John Stachel, *Science, New Series* **278**(5341) (November 14, 1997), 1270–1273.

Note. The author's of the *Science* paper state in the abstract of their work that "According to the commonly accepted view, David Hilbert completed the general theory of relativity at least 5 days before Albert Einstein submitted his conclusive paper on this theory on 25 November 1915." It has even been proposed that Einstein got the final version of the field equations from Hilbert or his work! Digging through original drafts of manuscripts and letters of correspondence between Einstein and Hilbert, they lay out a detailed time line and offer an alternative interpretation of the events, all of which occurred in two or three weeks in November of 1915. First, Einstein submits a paper to the Prussian Academy of Sciences on November 11, 1915 (which appears in *Sitzungsber. Preuss. Akad. Wiss.* **799**, 1915), but he does not have the trace term in his statement of the field equations (this is the $-\frac{1}{2}g_{\mu\nu}T$ term mentioned above). Hilbert submits a manuscript on November 20, 1915, a revised version of which is published later on March 31, 1916 (*Nach. Ges. Wiss. Goettingen* **395**); but Hilbert's published version includes a correct statement of the field equations. This lead some to think that Einstein actually got the field equations from Hilbert. However, the author's of the *Science* paper found a copy of the publisher's proofs of the version of Hilbert's manuscript as it was initially submitted (stamped with a date of December 6, 1915):



Erste Korrektur meiner ersten Note.

I

Die Grundlagen der Physik.

(Erste Mitteilung.)

Von

David Hilbert.

Vorgelegt in der Sitzung vom 20. November 1915.

Die tiefgreifenden Gedanken und originellen Begriffsbildungen, vermöge derer Mie seine Elektrodynamik aufbaut, und die gewaltigen Problemstellungen von Einstein sowie dessen scharfsinnige zu ihrer Lösung ersonnenen Methoden haben der Untersuchung über die Grundlagen der Physik neue Wege eröffnet.

Ich möchte im Folgenden — im Sinne der axiomatischen Methode — aus drei einfachen Axiomen ein ~~neues~~ System von Grundgleichungen der Physik aufstellen, die von idealer Schönheit sind, und in denen, wie ich glaube, die Lösung der gestellten Probleme enthalten ist. Die genauere Ausführung sowie vor Allem die spezielle Anwendung meiner Grundgleichungen auf die fundamentalen Fragen der Elektrizitätslehre behalte ich späteren Mitteilungen vor. wesentlichen

Es seien w_s $(s = 1, 2, 3, 4)$ irgendwelche die Weltpunkte wesentlich eindeutig benennende Koordinaten, die sogenannten Weltparameter. Die das Geschehen in w_s charakterisierenden Größen seien:

1) die zehn von Einstein zuerst eingeführten Gravitationspotentiale $g_{\mu\nu}$ $(\mu, \nu = 1, 2, 3, 4)$ mit symmetrischem Tensorcharakter gegenüber einer beliebigen Transformation der Weltparameter w_s;

2) die vier elektrodynamischen Potentiale q_s mit Vektorcharakter im selben Sinne.

Das physikalische Geschehen ist nicht willkürlich, es gelten vielmehr zunächst folgende zwei Axiome:

In fact, you can see in the margin that Hilbert has written that the gravitational potentials, $g_{\mu\nu}$, were "first introduced by Einstein." Now Einstein has given a presentation, "The Field Equations of Gravitation," to the Prussian Academy on November 25, 1915 in which he gives the final version of his field equations and the work is published in *Sitzungsber. Preuss. Akad. Wiss.* **844** (1915). (The paper we have been quoting in class is the *Annalen der Physik* paper which Einstein submitted on March 20, 1916, according to the Abraham Pais biography *Subtle is the Lord…: The Science and Life of Albert Einstein*, Oxford University Press, 1982 [see Chapter 32].) The difference in dates of November 20 and November 25 accounts for the "5 days" mentioned in the abstract of the *Science* paper. This paper states: " To summarize: Initially, Hilbert did not give the explicit form of the field equations; then, after Einstein had published his field equations, Hilbert claimed that no calculation is necessary; finally, he conceded that one is. Taken together, this sequence suggests that knowledge of Einstein's result may have been crucial to Hilbert's introduction of the trace term into his field equations." It now seems that the physics community accepts November 25, 1915 as the date on which the field equations first appeared and that Einstein deserves credit for them. In fact, there was a bit of celebration on November 25, 2015 in commemoration of the centennial. (However, you can still Google "Relativity priority dispute" and you see that some of the argument is still ongoing.)

Note. It seems that Einstein did feel some animosity towards Hilbert upon learning of the Hilbert manuscript, but in a letter from Einstein to Hilbert dated December 20, 1915 it is clear that the two were back on good terms. Though Hilbert's 1916 paper does mention "the magnificent theory of general relativity established by Einstein," the author's of the *Science* paper conclude that: "If Hilbert had only altered the dateline to read 'submitted on 20 November 1915, revised on (any date after 2 December 1915, the date of publication of Einstein's conclusive paper),' no later priority question could have arisen." Though the fact that Einstein and Hilbert were able to work out any dispute so quickly and with little or no public display of anger indicates the level of professionalism of these two giants of early 20th century physics and math.

Note. We now return to Faber's material before moving on the the Schwarzschild solution in the next section.

Note. The text argues that in a weak static gravitational field, we need

$$g_{00} = 1 + 2\Phi. \qquad (135)$$

See pages 204-206 for the argument. We will need this result in the Schwarzschild solution of the next section.

Lemma III-4. For each μ,

$$g^{\lambda\beta}\frac{\partial g_{\lambda\beta}}{\partial x^\mu} = \frac{1}{g}\frac{\partial g}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}[\ln|g|].$$

Proof. Now

$$g_{\lambda\beta}g^{\beta\sigma} = \delta^\sigma_\lambda = \begin{cases} 0 \text{ if } \sigma \neq \lambda \\ 1 \text{ if } \sigma = \lambda. \end{cases}$$

Then (summing over β but not over λ) we have

$$\sum_{\beta=0}^{3} g_{\lambda\beta}gg^{\beta\lambda} = g_{\lambda\lambda}gg^{\lambda\lambda} = g \text{ for } \lambda = 0, 1, 2, 3. \qquad (136)$$

We can express determinants in terms of permutations of the indices (see my online Theory of Matrices [MATH 5090] notes Section 3.1. Basic Definitions and Notation). We have

$$g = \sum_{\pi\in S_4} \text{sign}(\pi)g_{0\,\pi(0)}g_{1\,\pi(1)}g_{2\,\pi(2)}g_{3\,\pi(3)} \qquad (137)$$

where π is a permutation of the set $\{0, 1, 2, 3\}$ and so π ranges over all elements of the symmetric group S_4 (which is a group of order $4! = 24$). Differentiating g with respect to x^μ gives (by the Product Rule)

$$g = \sum_{\pi\in S_4} \text{sign}(\pi)\frac{\partial g_{0\,\pi(0)}}{\partial x^\mu}g_{1\,\pi(1)}g_{2\,\pi(2)}g_{3\,\pi(3)} + \sum_{\pi\in S_4} \text{sign}(\pi)g_{0\,\pi(0)}\frac{\partial g_{1\,\pi(1)}}{\partial x^\mu}g_{2\,\pi(2)}g_{3\,\pi(3)}$$

$$+\sum_{\pi\in S_4}\mathrm{sign}(\pi)g_{0\,\pi(0)}g_{1\,\pi(1)}\frac{\partial g_{2\,\pi(2)}}{\partial x^\mu}g_{3\,\pi(3)}+\sum_{\pi\in S_4}\mathrm{sign}(\pi)g_{0\,\pi(0)}g_{1\,\pi(1)}g_{2\,\pi(2)}\frac{\partial g_{3\,\pi(3)}}{\partial x^\mu}$$

Now the first term on the right hand of the previous equation is similar in form to equation (137) for the determinant of

$$(g_{\mu\nu})=\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03}\\ g_{10} & g_{11} & g_{12} & g_{13}\\ g_{20} & g_{21} & g_{22} & g_{23}\\ g_{30} & g_{31} & g_{32} & g_{33}\end{pmatrix}.$$

If we consider the matrix

$$\begin{pmatrix} \frac{\partial g_{00}}{\partial x^\mu} & \frac{\partial g_{01}}{\partial x^\mu} & \frac{\partial g_{02}}{\partial x^\mu} & \frac{\partial g_{03}}{\partial x^\mu}\\ g_{10} & g_{11} & g_{12} & g_{13}\\ g_{20} & g_{21} & g_{22} & g_{23}\\ g_{30} & g_{31} & g_{32} & g_{33}\end{pmatrix}$$

then we see that the determinant, by (137), is

$$\sum_{\pi\in S_4}\mathrm{sign}(\pi)\frac{\partial g_{0\,\pi(0)}}{\partial x^\mu}g_{1\,\pi(1)}g_{2\,\pi(2)}g_{3\,\pi(3)}$$

and by (136) with $\lambda=0$ this equals $\displaystyle\sum_{\beta=0}^{3}\frac{\partial g_{0\,\beta}}{\partial x^\mu}gg^{\beta 0}$. Similarly, the other three terms on the right hand side of (137) are of a corresponding form for $\lambda=1,2,3$. So (138) can be written as $\dfrac{\partial g}{\partial x^\mu}=\dfrac{\partial g_{\lambda\beta}}{\partial x^\mu}gg^{\beta\lambda}$ or, since $g^{\lambda\beta}=g^{\beta\lambda}$ (as Faber states on page 208), $\dfrac{\partial g}{\partial x^\mu}=\dfrac{\partial g_{\lambda\beta}}{\partial x^\mu}gg^{\lambda\beta}$. This implies $\dfrac{1}{g}\dfrac{\partial g}{\partial x^\mu}=g^{\lambda\beta}\dfrac{\partial g_{\lambda\beta}}{\partial x^\mu}$ and, since $\dfrac{\partial}{\partial x^\mu}[\ln|g|]=\dfrac{1}{g}\dfrac{\partial g}{\partial x^\mu}$, the claim follows. ∎

Note. From the definition of the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ we get by setting $\nu=\lambda$:

$$\Gamma^\lambda_{\mu\lambda} = \frac{1}{2}g^{\lambda\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^\lambda}+\frac{\partial g_{\lambda\beta}}{\partial x^\mu}-\frac{\partial g_{\mu\lambda}}{\partial x^\beta}\right)$$

$$
\begin{aligned}
&= \frac{1}{2} g^{\lambda\beta} \frac{\partial g_{\lambda\beta}}{\partial x^\mu} \text{ since we are summing over all } \lambda \text{ and } \beta \\
&\quad \text{then the first and last partial derivatives cancel} \\
&= \frac{1}{2} \frac{\partial}{\partial x^\mu}[\ln |g|] = \frac{\partial}{\partial x^\mu}[\ln |g|^{1/2}] \text{ by Lemma III-4.}
\end{aligned}
$$

So the Ricci tensor now satisfies

$$
\begin{aligned}
R_{\mu\nu} &= \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\Gamma^\lambda_{\beta\lambda} \\
&= \frac{\Gamma^\lambda_{\mu\nu}}{\partial x^\nu} - \frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial x^\mu}[\ln |g|^{1/2}] + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\frac{\partial}{\partial x^\beta}[\ln |g|^{1/2}] \\
&\quad \text{based on the value of } \Gamma^\lambda_{\mu\lambda} \text{ given above} \\
&= \frac{\Gamma^\lambda_{\mu\nu}}{\partial x^\nu} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \frac{\partial^2}{\partial x^\lambda \partial x^\mu}[\ln |g|^{1/2}] - \Gamma^\beta_{\mu\nu}\frac{\partial}{\partial x^\beta}[\ln |g|^{1/2}].
\end{aligned}
$$

We can now resolve the apparent conflict between Einstein's value of $R_{\mu\nu}$ and out value. The resolution is that Einstein is taking $R_{\mu\nu}$ as

$$\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta}$$

and Einstein is taking $S_{\mu\nu}$ as

$$-\frac{\partial^2}{\partial x^\nu \partial x^\mu}[\ln \sqrt{g}] - \Gamma^\beta_{\mu\nu}\frac{\partial}{\partial x^\beta}[\ln \sqrt{g}].$$

He then has $G_{\mu\nu} = R_{\mu\nu} + S_{\mu\nu}$ (but he claims the existence of coordinates that make $S_{\mu\nu} = 0$). *We* are taking $R_{\mu\nu}$ as Einstein's $G_{\mu\nu}$ (and we do not use the existence of such coordinates as Einstein claims). This resolves the difference between Einstein's formulas and our's (well, within a difference of g and $|g|$ and a questionable manipulation of the complex logarithm).

Revised: 6/30/2019