3.8 The Schwarzschild Solution

Note. In this section, we consider the first analytic solution to the field equations. Surprisingly, the solution was given in a paper submitted January 13, 1916! This is work of Karl Schwarzschild, who quotes Einstein's presentation of November 18, 1915 to the Prussian Academy as having posed the problem. So Schwarzschild has figured this out and written it up in under 2 months! The solution appears as "On the Gravitational Field of a Mass Point according to Einsteins Theory" *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, Phys.-Math. Klasse* (1916), 189–196. An English translation is available online at: `arxiv.org/pdf/physics/9905030v1.pdf`.

Note. Karl Schwarzschild (1873-1916) earned his doctorate at the University of Munich in the 1890s. From 1901 to 1909 he was a professor at Göttingen where he interacted with Felix Klein, David Hilbert, and Hermann Minkowski. When hostilities broke out in August 1914, at the start of World War I, Schwarzschild volunteered for military service in the German army. He was stationed in Belgium (where he ran a weather station), France (where he did computations for artillery and missile trajectories), and Russia. In Russia, he wrote two papers on Einstein's relativity and one paper on quantum theory! Sadly, he also contracted an autoimmmune disease of the skin while in Russia. He returned home in March 1916 and died on May 11, 1916. Schwarzschild had sent a copy of his paper on a solution to the field equations to Einstein who replied "I had not expected that one could formulate the exact solution of the problem in such a simple way." This information (and the photo above) is from the MacTutor History of Mathematics archive at: `www-history.mcs.st-andrews.ac.uk/Biographies/Schwarzschild.html`.

Note. In this section, we solve Einstein's field equations for the gravitational field outside an isolated sphere of mass M assumed to be at rest at the (spatial) origin of our coordinate system.

Note. We convert to spherical coordinates ρ, φ, θ:

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi. \end{aligned}$$

In the event of flat spacetime, we have the Lorentz metric (as is shown in Exercise 3.8.1):

$$\begin{aligned} d\tau^2 &= dt^2 - dx^2 - dy^2 - dz^2 \\ &= dt^2 - d\rho^2 - \rho^2 d\varphi^2 - \rho^2 \sin^2\varphi \, d\theta^2. \qquad (144) \end{aligned}$$

Note. As the book says, "the derivation that follows is not entirely rigorous, but it does not have to be - as long as the resulting metric form is a solution to the field equations."

Note. We have a static gravitational field (i.e. independent of time) and it is spherically symmetric (i.e. independent of φ and θ) so we look for a metric form satisfying

$$d\tau^2 = U(\rho)dt^2 - V(\rho)d\rho^2 - W(\rho)(\rho^2 d\varphi^2 + \rho^2 \sin^2\varphi \, d\theta^2) \qquad (145)$$

where U, V, W are functions of ρ only. Let $r = \rho\sqrt{W(\rho)}$ then (145) becomes

$$d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\varphi^2 - r^2 \sin^2\varphi \, d\theta^2 \qquad (146)$$

for some $A(r)$ and $B(r)$. Next define functions $m = m(r)$ and $n = n(r)$ where

$$A(r) = e^{2m(r)} = e^{2m} \text{ and } B(r) = e^{2n(r)} = e^{2n}.$$

Then (146) becomes

$$d\tau^2 = e^{2m}dt^2 - e^{2n}dr^2 - r^2 d\varphi^2 - r^2 \sin^2\varphi \, d\theta^2. \qquad (147)$$

Since $d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$ in general, if we label $x^0 = t$, $x^1 = r$, $x^2 = \varphi$, $x^3 = \theta$ we have

$$(g_{\mu\nu}) = \begin{pmatrix} e^{2m} & 0 & 0 & 0 \\ 0 & -e^{2n} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\varphi \end{pmatrix}$$

and $g = \det(g_{ij}) = -e^{2m+2n}r^4\sin^2\varphi$. If we find $m(r)$ and $n(r)$, we will have a solution!

Note. We need the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right). \qquad (126)$$

Since $g_{\mu\nu} = 0$ for $\mu \neq \nu$, we have $g^{\mu\mu} = 1/g_{\mu\mu}$ and $g^{\mu\nu} = 0$ if $\mu \neq \nu$. So the coefficient $g^{\lambda\beta}$ is 0 unless $\beta = \lambda$ and we have

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right).$$

We need to consider three cases:

Case 1. For $\lambda = \nu$:

$$\begin{aligned}\Gamma^{\nu}_{\mu\nu} &= \frac{1}{2g_{\nu\nu}}\left(\frac{\partial g_{\mu\nu}}{\partial x^{\nu}} + \frac{\partial g_{\nu\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\nu}}\right)\\ &= \frac{1}{2g_{\nu\nu}}\left(\frac{\partial g_{\nu\nu}}{\partial x^{\mu}}\right) = \frac{1}{2}\frac{\partial}{\partial x^{\mu}}[\ln(g_{\nu\nu})]\end{aligned}$$

Case 2. For $\mu = \nu \neq \lambda$:

$$\begin{aligned}\Gamma^{\lambda}_{\mu\mu} &= \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\mu\lambda}}{\partial x^{\mu}} + \frac{\partial g_{\mu\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\mu\mu}}{\partial x^{\lambda}}\right)\\ &= \frac{-1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\mu\mu}}{\partial x^{\lambda}}\right)\end{aligned}$$

since $g_{\mu\lambda} = 0$ in this case.

Case 3. For μ, ν, λ distinct: $\Gamma^{\lambda}_{\mu\nu} = 0$ since $g_{\mu\lambda} = g_{\nu\lambda} = g_{\mu\nu} = 0$ in this case.

Note. With the $g_{\mu\nu}$'s given above (in terms of m, n, r and φ) we can calculate the nonzero Christoffel symbols to be:

$$\begin{array}{ll}\Gamma^0_{10} = \Gamma^0_{01} = m' & \Gamma^1_{00} = m'e^{2m-2n}\\ \Gamma^1_{11} = n' & \Gamma^1_{22} = -re^{-2n}\\ \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} & \Gamma^1_{33} = -re^{-2n}\sin^2\varphi\\ \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} & \Gamma^3_{23} = \Gamma^3_{32} = \cot\varphi\\ & \Gamma^2_{33} = -\sin\varphi\cos\varphi\end{array}$$

where $' = d/dr$.

Note. We have

$$\ln|g|^{1/2} = \frac{1}{2}\ln(e^{2m+2n}r^4\sin^2\varphi) = m + n + 2\ln r + \ln(\sin\varphi).$$

We saw in Lemma III-4 that

$$g^{\lambda\beta}\frac{\partial g_{\lambda\beta}}{\partial x^\mu} = \frac{1}{g}\frac{\partial g}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}[\ln|g|].$$

Now

$$\frac{\partial}{\partial x^\beta}[\ln|g|^{1/2}] = \frac{1}{2}\frac{\partial}{\partial x^\beta}[\ln|g|] = \frac{1}{2}g^{\lambda\mu}\frac{\partial g_{\lambda\mu}}{\partial x^\beta} = \frac{1}{2}g^{\lambda\lambda}\frac{\partial g_{\lambda\lambda}}{\partial x^\beta}.$$

Also, from (126)

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right)$$

we have with $\mu = \beta$, $\nu = \lambda$ and δ the dummy variable:

$$\begin{aligned}
\Gamma^\lambda_{\beta\lambda} &= \frac{1}{2}g^{\lambda\delta}\left(\frac{\partial g_{\beta\delta}}{\partial x^\lambda} + \frac{\partial g_{\lambda\delta}}{\partial x^\beta} - \frac{\partial g_{\beta\lambda}}{\partial x^\delta}\right) \\
&= \frac{1}{2}g^{\lambda\lambda}\left(\frac{\partial g_{\beta\lambda}}{\partial x^\lambda} + \frac{\partial g_{\lambda\lambda}}{\partial x^\beta} - \frac{\partial g_{\beta\lambda}}{\partial x^\lambda}\right) \\
&= \frac{1}{2}g^{\lambda\lambda}\left(\frac{\partial g_{\lambda\lambda}}{\partial x^\beta}\right).
\end{aligned}$$

Therefore we have

$$\frac{\partial}{\partial x^\beta}[\ln|g|^{1/2}] = \Gamma^\lambda_{\beta\lambda}.$$

Similarly $\frac{\partial}{\partial x^\mu}[\ln|g|^{1/2}] = \Gamma^\lambda_{\mu\lambda}$. Therefore the field equations imply

$$R_{\mu\nu} = \frac{\partial^2}{\partial x^\mu \partial x^\nu}[\ln|g|^{1/2}] - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\beta_{\mu\nu}\frac{\partial}{\partial x^\beta}[\ln|g|^{1/2}] = 0.$$

Note. We find that

$$\begin{aligned}
R_{00} &= \left(-m'' + m'n' - m'^2 - \frac{2m'}{r}\right)e^{2m-2n} \\
R_{11} &= m'' - m'n' + m'^2 - \frac{2n'}{r} \\
R_{22} &= e^{-2n}(1 + rm' - rn') - 1 \\
R_{33} &= R_{22}\sin^2\varphi
\end{aligned}$$

All other $R_{\mu\nu}$ are identically zero. Next, the field equations say that we need each of these to be zero. Therefore we need:

$$\begin{aligned}\left(-m'' + m'n' - m'^2 - \frac{2m'}{r}\right) &= 0\\ m'' - m'n' + m'^2 - \frac{2n'}{r} &= 0\\ e^{-2n}(1 + rm' - rn') - 1 &= 0\\ R_{22}\sin^2\varphi &= 0\end{aligned}$$

Adding the first two of these equations, we find that $m' + n' = 0$, and so $m + n = b$, a constant. However, **by the boundary conditions** both m and n must vanish as $r \to \infty$, since the metric (147) must approach the Lorentz metric at great distances from the mass M (compare (147) and (144)). Therefore, $b = 0$ and $n = -m$ The third equation implies:

$$1 = (1 + 2rm')e^{2m} = (re^{2m})'.$$

Hence we have $re^{2m} = r + C$ for some constant C, or $g_{00} = e^{2m} = 1 + C/r$. *But as commented in the previous section, we need $g_{00} = 1 - 2M/r$ where the field is weak.* We therefore have $C = -2M$. Hence we have the solution:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2 d\varphi^2 - r^2\sin^2\varphi\, d\theta^2.$$

Note. Notice that this solution has two singularities. One at the center of the mass, $r = 0$, and another at $r = 2M$. This second singularity will correspond to the event horizon when we address black holes. Also, we have $r = \rho$ and $W(\rho) = 1$.

Revised: 6/30/2019