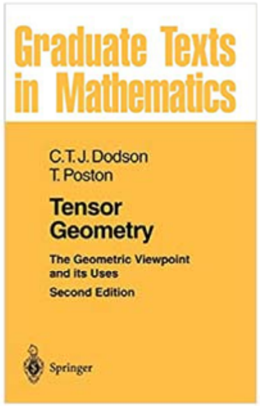
Differential Geometry

Chapter II. Affine Spaces

II.1. Spaces—Proofs of Theorems



Lemma II.1.A

Lemma II.1.A. In an affine space with difference function $\mathbf{d}$ we have

(a) $\mathbf{d}(\mathbf{x},\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in X$, and

(b) $\mathbf{d}(\mathbf{x},\mathbf{y}) = -\mathbf{d}(\mathbf{y},\mathbf{x})$ for all $\mathbf{x},\mathbf{y} \in X$.

Proof. (a) Let $\mathbf{x} \in X$. Then by Definition II.1.01 (A i) with $\mathbf{x} = \mathbf{y} = \mathbf{z}$ we have $\mathbf{d}(\mathbf{x},\mathbf{x}) + \mathbf{d}(\mathbf{x},\mathbf{x}) = \mathbf{d}(\mathbf{x},\mathbf{x})$, or $\mathbf{d}(\mathbf{x},\mathbf{x}) = \mathbf{0}$, as claimed.

(b) Let $\mathbf{x},\mathbf{y} \in X$. Then by Definition II.1.02 (A i) with $\mathbf{z} = \mathbf{x}$ we have $\mathbf{d}(\mathbf{x},\mathbf{y}) + \mathbf{d}(\mathbf{y},\mathbf{x}) = \mathbf{d}(\mathbf{x},\mathbf{x})$ and so by part (a), $\mathbf{d}(\mathbf{x},\mathbf{y}) + \mathbf{d}(\mathbf{y},\mathbf{x}) = \mathbf{0}$ or $\mathbf{d}(\mathbf{x},\mathbf{y}) = -\mathbf{d}(\mathbf{y},\mathbf{x})$, as claimed. □

Lemma II.1.05

Lemma II.1.05. Two affine subspaces X' and X'' of X are parallel if and only if $X'' = X' + \mathbf{t}$ for some $\mathbf{t} \in T$.

Proof. (1) Suppose X' and X'' are parallel. Let $x' \in X'$ and $x'' \in X''$ be fixed.

Suppose $y'' \in X''$. Set $\mathbf{t} = \mathbf{d}(x',x'')$. Then $\mathbf{d}(x'',y'') \in \mathbf{d}(X'' \times X'')$ and so $\mathbf{d}(x'',y'') \in \mathbf{d}(X',X')$ since $\mathbf{d}(X' \times X') = \mathbf{d}(X'' \times X'')$ because X' and X'' are parallel by hypothesis. Since $x',x'',y'' \in X$ then by Definition II.1.01 (A i) $\mathbf{d}(x',y'') = \mathbf{d}(x',x'') + \mathbf{d}(x'',y'')$. We have set $\mathbf{t} = \mathbf{d}(x',x'') \in T$ and we have $\mathbf{s} = \mathbf{d}(x'',y'') \in \mathbf{d}(X' \times X')$, so $\mathbf{d}(x',y'') = \mathbf{t} + \mathbf{s}$ where $\mathbf{s} \in \mathbf{d}(X' \times X')$ and $\mathbf{t} \in T$. By Note II.1.A, we then have $y'' = x' + \mathbf{t} + \mathbf{s}$ where $\mathbf{t} \in T$ and $\mathbf{s} \in \mathbf{d}(X' \times X')$. So by Exercise II.1.2(b), $y'' \in X' + \mathbf{t}$. Therefore $X'' \subseteq X' + \mathbf{t}$.

Lemma II.1.05 (continued 1)

Proof (continued). Now suppose $y'' \in X' + \mathbf{t}$ where $\mathbf{t} = \mathbf{d}(x',x'')$. By Exercise II.1.2(b), $y'' = x' + \mathbf{t} + \mathbf{s}$ for some $\mathbf{s} \in \mathbf{d}(X' \times X')$ (and this holds for any element of X', so we use $x' \in X'$ from above). This means, by Note II.1. A, that $\mathbf{d}(x',y'') = \mathbf{t} + \mathbf{s} = \mathbf{d}(x',x'') + \mathbf{s}$. By Definition II.1.01 (A i) we have $\mathbf{d}(x',y'') = \mathbf{d}(x',x'') + \mathbf{d}(x'',y'')$, so that $\mathbf{s} = \mathbf{d}(x'',y'') \in \mathbf{d}(X' \times X')$ and since $\mathbf{d}(X' \times X') = \mathbf{d}(X'' \times X'')$ by the parallel hypothesis, then $\mathbf{d}(x'',y'') \in \mathbf{d}(X'' \times X'')$. Now X'' is an affine subspace with vector space $\mathbf{d}(X'' \times X'')$ (by the definition of subspace). Since $x'' \in X''$ then $\mathbf{d}_{x''} : \{x''\} \times X'' \to \mathbf{d}(X'' \times X'')$ is a bijection by Definition II.1.01 (A ii). Since $\mathbf{d}(x'',y'') \in \mathbf{d}(X'' \times X'')$ then for some $z'' \in X''$ we have $\mathbf{d}_{x''}(x'',z'') \in \mathbf{d}(x'',y'')$ since $\mathbf{d}_{x''}$ is onto (surjective). But $\mathbf{d}_{x''} : \{x''\} \times X \to T$ is also a bijection by Definition II.1.01 (A ii) and so is one to one (injective); since $\mathbf{d}_{x''}(x'',z'') = \mathbf{d}(x'',y'') = \mathbf{d}_{x''}(x'',y'')$ then $z'' = y''$ and so $y'' \in X''$. Therefore $X' + \mathbf{t} \subseteq X''$ and combining this with the above result, we have $X'' = X' + \mathbf{t}$, as claimed.

Lemma II.1.05 (continued 2)

Proof (continued). (2) Now suppose $X'' = X' + \mathbf{t}$. Let x'' and y'' be arbitrary elements of X''. Since $x'' \in X' + \mathbf{t}$ then $x'' = x' + \mathbf{t}$ for some $x' \in X'$. By Note II.1.A this means $\mathbf{d}(x', x'') = \mathbf{t}$ or $\mathbf{d}_{x'}(x', x'') = \mathbf{t}$ and, similarly, $\mathbf{d}(y', y'') = \mathbf{t}$ or $\mathbf{d}_{y'}(y', y'') = \mathbf{t}$ for some $y' \in X'$. Then

$$\begin{aligned} \mathbf{d}(x', y') &= \mathbf{d}(x', x'') + \mathbf{d}(x'', y'') + \mathbf{d}(y'', y') \text{ by Definition II.1.01 (A i)} \\ &= \mathbf{t} + \mathbf{d}(x'', y'') - \mathbf{d}(y', y'') \text{ by Lemma II.1.A(b)} \\ &= \mathbf{t} + \mathbf{d}(x'', y'') - \mathbf{t} = \mathbf{d}(x''). \end{aligned}$$

Since $\mathbf{d}(x', y') \in \mathbf{d}(X' \times X')$ then $\mathbf{d}(x'', y'') \in \mathbf{d}(X' \times X')$ and, since $x'', y'' \in X''$ are arbitrary, then $\mathbf{d}(X'' \times X'') \subseteq \mathbf{d}(X' \times X')$. Similarly, if $x', y' \in X''$ then $\mathbf{d}(x', y') = \mathbf{d}(x'', y'') \in \mathbf{d}(X'' \times X'')$ and $\mathbf{d}(X' \times X') \subseteq \mathbf{d}(X'' \times X'')$. Therefore, $\mathbf{d}(X' \times X') = \mathbf{d}(X'' \times X'')$ and X' and X'' are parallel, as claimed. □

Lemma II.1.06

Lemma II.1.06. For X a vector space, $X' \subseteq X$ is an affine subspace of X if and only if X' is a translate of some vector subspace of X.

Proof. Now X' is a *set* of vectors from vector space X. So we use the natural affine structure of X and have $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \mathbf{x}$. So $\mathbf{d}(X' \times X') = \{\mathbf{y} - \mathbf{x} \mid \mathbf{x}, \mathbf{y} \in X'\}$. By the definition of affine subspace (Definition II.1.03(i)) $\mathbf{d}(X' \times X')$ is a vector subspace of X, denote $X'' = \mathbf{d}(X' \times X')$. Now $\mathbf{d}(X'' \times X'')$ is the difference of all vectors in X'' so $\mathbf{d}(X'' \times X'') \subseteq X''$ since X'' is a vector space. Also, since $\mathbf{0} \in X''$ then $\mathbf{d}(X'' \times X'' \subseteq X''$ and hence $\mathbf{d}(X'' \times X'') \subseteq X''$ since X'' is a vector space. Also, since $\mathbf{0} \in X''$ then $\mathbf{d}(X'' \times X'') \supseteq X''$ and hence $\mathbf{d}(X'' \times X'') = X''$. Hence $X'' = \mathbf{d}(X' \times X') = \mathbf{d}(X'' \times X'')$. Since X'' is a vector subspace of X, then X'' is an affine subspace of X (by the definition of affine subspace). So by definition, X' and X'' are parallel affine subspaces of X. Hence, by Lemma II.1.05, X' is a translate of vector subspace X'' of X, as claimed.

Lemma II.1.06 (continued)

Lemma II.1.06. For X a vector space, $X' \subseteq X$ is an affine subspace of X if and only if X' is a translate of some vector subspace of X.

Proof (continued). Conversely, if X' is a translate of an affine subspace of X then by Exercise II.1.2(c) X' is an affine subspace of X. □