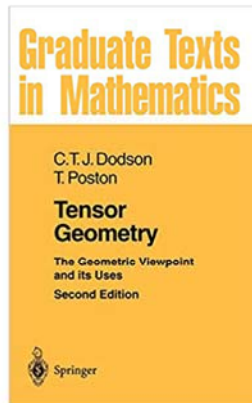


# Differential Geometry

## Chapter III. Dual Spaces

### III.1. Contours, Covariance, Contravariance, Dual Basis—Proofs of Theorems



## Lemma III.1.04

**Lemma III.1.04.** Let  $X$  be an  $n$ -dimensional real vector space with dual space  $X^*$ . Then  $\dim(X^*) = \dim(X)$ .

**Proof.** Let  $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$  be a basis for  $X$ . For  $\mathbf{x} \in X$  where  $\mathbf{x} = a^1\mathbf{b}_1 + a^2\mathbf{b}_2 + \dots + a^n\mathbf{b}_n = a^i\mathbf{b}_i$ , define the  $n$  linear functionals

$$\mathbf{b}^i : X \rightarrow \mathbb{R} \text{ as } \mathbf{b}^i(\mathbf{x}) = a^i \text{ for } i = 1, 2, \dots, n.$$

For any linear functional  $\mathbf{f} \in X^*$ , there is a matrix  $[\mathbf{f}]$  representing  $\mathbf{f}$  based on bases  $\beta$  of  $X$  and basis  $\{\mathbf{e}_i\}$  of  $\mathbb{R}$  (see "Theorem 3.10. Matrix Representation of Linear Transformations" in my online notes for 3.4. **Linear Transformations** for Linear Algebra [MATH 2010]). Notice that  $[\mathbf{f}]$  is  $q \times n$ . Let the  $j$ th column of  $[\mathbf{f}]$  be the scalar  $f_j^1$  so that  $[\mathbf{f}] = [f_1^1, f_2^1, \dots, f_n^1]$ .

## Lemma III.1.04 (continued)

**Lemma III.1.04.** Let  $X$  be an  $n$ -dimensional real vector space with dual space  $X^*$ . Then  $\dim(X^*) = \dim(X)$ .

**Proof (continued).** Then

$$\begin{aligned} [\mathbf{f}] &= [f_1^1, f_2^1, \dots, f_n^1] \\ &= f_1^1[1, 0, \dots, 0] + f_2^1[0, 1, 0, \dots, 0] + \dots + f_n^1[0, 0, \dots, 0, 1] \\ &= f_1^1[\mathbf{b}^1] + f_2^1[\mathbf{b}^2] + \dots + f_n^1[\mathbf{b}^n] \text{ since the } 1 \times n \text{ matrices in the} \\ &\quad \text{previous line represent the functionals in } \beta \\ &= f_j^1[\mathbf{b}^j] \text{ using the Einstein summation convention} \\ &= [f_j^1 \mathbf{b}^j]. \end{aligned}$$

Now the choice of the  $f_j^1$  for  $j = 1, 2, \dots, n$  is unique since  $f_j^1 = \mathbf{f}(\mathbf{b}_j)$  for  $j = 1, 2, \dots, n$ . So the representation of  $\mathbf{F} \in X^*$  as a linear combination of  $\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^n$  is unique and hence  $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^n\}$  is a basis for  $X^*$ . Therefore  $\dim(X^*) = n = \dim(X)$ .  $\square$

## Lemma III.1.A

**Lemma III.1.A.** Given a linear functional  $\mathbf{f} \in X^*$  where  $X = \mathbb{R}^n$ , there is  $\mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{f}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle$  (the inner product on  $\mathbb{R}^n$ ), and conversely for each  $\mathbf{y} \in \mathbb{R}^n$  the mapping  $\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$  is a linear functional in  $X^*$ . That is,  $X^*$  is isomorphic to  $\mathbb{R}^n$  when  $X = \mathbb{R}^n$ .

**Proof.** Let  $\mathbf{f} \in X^* = (\mathbb{R}^n)^*$ . Let  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be the standard basis of  $X = \mathbb{R}^n$  and define  $\mathbf{y} \in \mathbb{R}^n$  as  $\mathbf{y} = \mathbf{f}(\mathbf{e}_1)\mathbf{e}_1 + \mathbf{f}(\mathbf{e}_2)\mathbf{e}_2 + \dots + \mathbf{f}(\mathbf{e}_n)\mathbf{e}_n$ . Then for  $\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + \dots + x^n\mathbf{e}_n$  we have

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \mathbf{f}(x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + \dots + x^n\mathbf{e}_n) \\ &= x^1\mathbf{f}(\mathbf{e}_1) + x^2\mathbf{f}(\mathbf{e}_2) + \dots + x^n\mathbf{f}(\mathbf{e}_n) \text{ since } \mathbf{f} \text{ is linear} \\ &= \langle [x^1, x^2, \dots, x^n], [\mathbf{f}(\mathbf{e}_1), \mathbf{f}(\mathbf{e}_2), \dots, \mathbf{f}(\mathbf{e}_n)] \rangle \\ &= \langle \mathbf{x}, \mathbf{y} \rangle, \end{aligned}$$

as claimed.

## Lemma III.1.A (continued)

**Lemma III.1.A.** Given a linear functional  $\mathbf{f} \in X^*$  where  $X = \mathbb{R}$ , there is  $\mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{f}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle$  (the inner product on  $\mathbb{R}^n$ ), and conversely for each  $\mathbf{y} \in \mathbb{R}^n$  the mapping  $\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$  is a linear functional in  $X^*$ . That is,  $X^*$  is isomorphic to  $\mathbb{R}^n$  when  $X = \mathbb{R}^n$ .

**Proof (continued).** Conversely, for given  $\mathbf{y} \in \mathbb{R}^n$ , the mapping  $\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$  is a functional and is linear since for scalars  $a, b \in \mathbb{R}$  we have  $\langle a\mathbf{x} + b\mathbf{z}, \mathbf{y} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle + b\langle \mathbf{z}, \mathbf{y} \rangle$ . So the mapping  $\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$  is a linear functional for each  $\mathbf{y} \in \mathbb{R}^n$ . Therefore  $X^* = (\mathbb{R}^n)^* \cong \mathbb{R}^n = X$  under the vector space isomorphism  $\mathbf{f} \mapsto \langle \cdot, \mathbf{y} \rangle$  where  $\mathbf{f}\mathbf{x} = \langle \mathbf{x}, \mathbf{y} \rangle$ .  $\square$

## Theorem III.1.A

**Theorem III.1.A.** Let  $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$  be a basis for  $X$  and  $\beta' = \{\mathbf{b}'_1, \mathbf{b}'_2, \dots, \mathbf{b}'_m\}$  be a basis for  $Y$  such that the  $m \times n$  matrix  $A = [\mathbf{A}]_{\beta}^{\beta'}$  represents a linear transformation from  $X$  to  $Y$  with respect to ordered bases  $\beta$  and  $\beta'$ . Let  $\beta^*$  and  $\beta'^*$  be the dual bases of  $X^*$  and  $Y^*$ , respectively. Then the  $n \times m$  matrix  $A^* = [\mathbf{A}^*]_{\beta'^*}^{\beta^*} = ([\mathbf{A}]_{\beta}^{\beta'})^t$ , where  $t$  represents the transpose operator on a matrix.

**Proof.** Let  $\mathbf{f} \in Y^*$  where  $\mathbf{f} = f_1\mathbf{b}'^1 + f_2\mathbf{b}'^2 + \dots + f_m\mathbf{b}'^m$  where  $\beta'^* = \{\mathbf{b}'^1, \mathbf{b}'^2, \dots, \mathbf{b}'^m\}$  is the dual basis of  $Y^*$ , so that  $\mathbf{f} = f_j\mathbf{b}'^j$  with the Einstein summation convention. Then  $A^*\mathbf{f} = \mathbf{A}^*(f_j\mathbf{b}'^j) \in X^*$ . Hence applying these functionals in  $X^*$  to the elements of basis  $\beta$  of  $X$  gives...

## Theorem III.1.A (continued 1)

**Proof (continued).**

$$\begin{aligned} (\mathbf{A}^*\mathbf{F})\mathbf{b}_i &= (\mathbf{A}^*(f_j\mathbf{b}'^j) = f_j(\mathbf{A}^*\mathbf{b}'^j)\mathbf{b}_i \text{ since scalars commute} \\ &= f_j(\mathbf{b}'^j\mathbf{A})\mathbf{b}_i \text{ by the definition of dual map } \mathbf{A}^* \text{ (here } \mathbf{f} \in Y^* \\ &\quad \text{of the definition is replaced with } \mathbf{b}'^j \in \beta'^j \subset Y^*) \\ &= f_j\mathbf{b}'^j(\mathbf{A}\mathbf{b}_i) \text{ by associativity} \\ &= f_j\mathbf{b}'^j(A(0, 0, \dots, 0, 1, 0, \dots, 0)^t) \text{ since the coordinate vector} \\ &\quad \text{of } \mathbf{b}_i \text{ with respect to the ordered basis } \beta \text{ is} \\ &\quad (0, 0, \dots, 0, 1, 0, \dots, 0) \text{ and matrix } A \text{ represents} \\ &\quad \mathbf{A} : X \rightarrow Y \text{ with respect to ordered bases } \beta \text{ and } \beta' \\ &= f_j\mathbf{b}'^j(a_i^1, a_i^2, \dots, a_i^m)^t \text{ since the matrix product produces the} \\ &\quad i\text{th column of } A, \text{ which is a vector in } Y \text{ and this column} \\ &\quad \text{vector is in coordinates with respect to ordered basis} \\ &\quad \beta' = \{\mathbf{b}'_1, \mathbf{b}'_2, \dots, \mathbf{b}'_m\} \text{ of } Y \dots \end{aligned}$$

## Theorem III.1.A (continued 2)

**Proof (continued).**

$$\begin{aligned} (\mathbf{A}^*\mathbf{F})\mathbf{b}_i &= f_j\mathbf{b}'^j(a_i^1, a_i^2, \dots, a_i^m)^t \\ &= f_j\mathbf{a}'^j_i \text{ by the definition of dual basis vector } \mathbf{b}'^j : Y \rightarrow \mathbb{R} \\ &\quad \text{(or since the coordinate vector in the previous line represents} \\ &\quad \mathbf{a}_i^k\mathbf{b}'_k \text{ and } \mathbf{b}'^j\mathbf{b}'_k = \delta_k^j \text{ by Note III.1.A).} \quad (*) \end{aligned}$$

Now  $\mathbf{A}^*\mathbf{f} : X \rightarrow \mathbb{R}$ , so  $\mathbf{A}^*\mathbf{f} \in X^*$  and can be written in terms of the dual basis  $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^n\}$ . In doing so, say  $\mathbf{A}^*\mathbf{f} = c_k\mathbf{b}^k$ , we have by (\*) that

$$f_j\mathbf{a}'^j_i = (\mathbf{A}^*\mathbf{f})\mathbf{b}_i = (c_k\mathbf{b}^k)\mathbf{b}_i = c_k\delta_i^k \text{ (by Note III.1.A)}$$

so that  $c_i = f_j\mathbf{a}'^j_i$  for  $i = 1, 2, \dots, n$ . Hence  $\mathbf{A}^*\mathbf{f} = f_j\mathbf{a}'^j_i\mathbf{b}^i$  and the coordinate vector of  $\mathbf{A}^*\mathbf{f}$  with respect to ordered basis  $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^n\}$  is  $(f_j\mathbf{a}'^j_1, f_j\mathbf{a}'^j_2, \dots, f_j\mathbf{a}'^j_n)$ . But applying  $A^t$  to  $\mathbf{f} = (f_1, f_2, \dots, f_m)$  (the coordinate vector of  $\mathbf{f} \in Y^*$  with respect to dual basis  $\beta'^* = \{\mathbf{b}'^1, \mathbf{b}'^2, \dots, \mathbf{b}'^m\}$ ) gives...

## Theorem III.1.A (continued 3)

Proof (continued).

$$\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{pmatrix}^T \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} \\ = \begin{pmatrix} a_1^1 f_1 + a_2^1 f_2 + \cdots + a_n^1 f_m \\ a_2^1 f_1 + a_2^2 f_2 + \cdots + a_2^m f_m \\ \vdots \\ a_n^1 f_1 + a_n^2 f_2 + \cdots + a_n^m f_m \end{pmatrix} = \begin{pmatrix} f_j a_j^1 \\ f_j a_j^2 \\ \vdots \\ f_j a_j^n \end{pmatrix}.$$

Therefore, the  $n \times m$  matrix  $A^t = ([\mathbf{A}]_{\beta}^{\beta'})^t$  has the same effect as the dual map  $\mathbf{A}^* : Y^* \rightarrow X^*$  with respect to the ordered bases  $\beta'^*$  and  $\beta^*$ . That is,  $[\mathbf{A}^*]_{\beta'^*}^{\beta^*} = ([\mathbf{A}]_{\beta}^{\beta'})^t$ , as claimed.  $\square$

## Theorem III.1.B

**Theorem III.1.B.** Let  $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$  and  $\beta' = \{\mathbf{b}'_1, \mathbf{b}'_2, \dots, \mathbf{b}'_n\}$  be bases for  $X$  and let  $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^n\}$  be the dual basis of  $\beta$  (so that  $\beta^*$  is a basis of  $X^*$ ). With  $\beta'^* = \{\mathbf{b}'^1, \mathbf{b}'^2, \dots, \mathbf{b}'^n\}$  the dual basis of  $\beta'$ , for  $\mathbf{f} \in X^*$  where

$$\mathbf{f} = \sum_{i=1}^n f_i \mathbf{b}^i = f_i \mathbf{b}^i = \sum_{i=1}^n f'_i \mathbf{b}'^i = f'_i \mathbf{b}'^i$$

we have  $f'_i = b_j^i f_j$  where the  $b_j^i$  are coordinates of  $\mathbf{b}'_i \in \beta'$  with respect to ordered basis  $\beta$  (that is,  $\mathbf{b}'_i$  satisfies  $\mathbf{b}'_i = \sum_{j=1}^n b_j^i \mathbf{b}_j = b_j^i \mathbf{b}_j$ ).

**Proof.** The matrix which converts coordinate vectors with respect to ordered basis  $\beta$  to coordinate vectors with respect to ordered basis  $\beta'$  is ...

## Theorem III.1.B (continued 1)

Proof (continued).

$$[\mathbf{I}]_{\beta'}^{\beta'} = ([\mathbf{I}]_{\beta'}^{\beta})^{-1} = \begin{bmatrix} b_1^1 & b_2^1 & \cdots & b_n^1 \\ b_1^2 & b_2^2 & \cdots & b_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1^n & b_2^n & \cdots & b_n^n \end{bmatrix}^{-1}$$

since  $\mathbf{b}'_i = b_j^i \mathbf{b}_j$  by the previous Note (here  $\beta$  and  $\beta'$  are interchanged from the Note). By Theorem III.1.A, the dual map of  $\mathbf{I} : X \rightarrow X$  (here  $\mathbf{I} \in L(X, X)$  maps each vector to itself but matrix  $[\mathbf{I}]_{\beta}^{\beta'}$  allows us to represent this map as a conversion of coordinate vectors with respect to  $\beta$  to coordinate vectors with respect to  $\beta'$ ),  $\mathbf{I}^* : X^* \rightarrow X^*$  (satisfying  $\mathbf{I}^*(\mathbf{f}) = \mathbf{f} \circ \mathbf{I}$  for all  $\mathbf{f} \in X^*$ , by the definition of dual map), has matrix representation...

## Theorem III.1.B (continued 2)

Proof (continued).

$$[\mathbf{I}^*]_{\beta'^*}^{\beta'^*} = ([\mathbf{I}]_{\beta'}^{\beta})^T = \begin{bmatrix} b_1^1 & b_2^1 & \cdots & b_n^1 \\ b_1^2 & b_2^2 & \cdots & b_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1^n & b_2^n & \cdots & b_n^n \end{bmatrix}.$$

So for  $\mathbf{f} \in X^*$  where

$$\mathbf{f} = \sum_{i=1}^n f_i \mathbf{b}^i = f_i \mathbf{b}^i = \sum_{i=1}^n f'_i \mathbf{b}'^i = f'_i \mathbf{b}'^i$$

(so that the coordinate [row] vectors of  $\mathbf{f}$  are  $(f_1, f_2, \dots, f_n)$  and  $(f'_1, f'_2, \dots, f'_n)$  with respect to ordered bases  $\beta^*$  and  $\beta'^*$ , respectively), we have...

## Theorem III.1.B (continued 3)

**Proof (continued).**

$$\begin{bmatrix} f'_1 \\ f'_2 \\ \vdots \\ f'_n \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ b_2^1 & b_2^2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} b_1^j f_j \\ b_2^j f_j \\ \vdots \\ b_n^j f_j \end{bmatrix},$$

so that  $f'_i = b_i^j f_j$ , as claimed.  $\square$