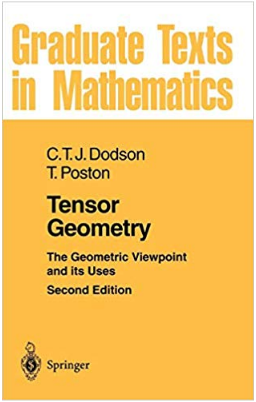
Differential Geometry

Chapter III. Dual Spaces

III.1. Contours, Covariance, Contravariance, Dual Basis—Proofs of Theorems



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Lemma III.1.04. Let X be an n-dimensional real vector space with dual space X^*. Then $\dim(X^*) = \dim(X)$.

Proof. Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for X. For $\mathbf{x} \in X$ where $\mathbf{x} = a^1\mathbf{b}_1 + a^2\mathbf{b}_2 + \cdots a^n\mathbf{b}_n = a^i\mathbf{b}_i$, define the n linear functionals

$$\mathbf{b}^i : X \to \mathbb{R} \text{ as } \mathbf{b}^i(\mathbf{x}) = a^i \text{ for } i = 1, 2, \ldots, n.$$

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$$\mathbf{b}^i : X \to \mathbb{R} \text{ as } \mathbf{b}^i(\mathbf{x}) = a^i \text{ for } i = 1, 2, \ldots, n.$$

For any linear functional $\mathbf{f} \in X^*$, there is a matrix $[\mathbf{f}]$ representing $\mathbf{f}$ based on bases β of X and basis $\{\mathbf{e}_i\}$ of $\mathbf{R}$ (see "Theorem 3.10. Matrix Representation of Linear Transformations" in my online notes for 3.4. Linear Transformations for Linear Algebra [MATH 2010]). Notice that $[\mathbf{f}]$ is $q \times n$. Let the jth column of $[\mathbf{f}]$ be the scalar f_j^1 so that $[\mathbf{f}] = [f_1^1, f_2^1, \ldots, f_n^1]$.

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Lemma III.1.04 (continued)

Lemma III.1.04. Let X be an n-dimensional real vector space with dual space X^*. Then $\dim(X^*) = \dim(X)$.

Proof (continued). Then

$$\begin{aligned} [\mathbf{f}] &= [f_1^1, f-2^1, \ldots, f_n^1] \\ &= f_1^1[1,0,\ldots,0] + f_2^1[0,1,0,\ldots,0] + \cdots + f_n^1[0,0,\ldots,0,1] \\ &= f_1^1[\mathbf{b}^1] + f_2^1[\mathbf{b}^2] + \cdots + f_n^1[\mathbf{b}^n] \text{ since the } 1\times n \text{ matrices in the} \\ &\quad\ \text{previous line represent the functionals in } \beta \\ &= f_j^1[\mathbf{b}^j] \text{ using the Einstein summation convention} \\ &= [f_j^1\mathbf{b}^j]. \end{aligned}$$

Now the choice of the f_j^1 for $j = 1, 2, \ldots, n$ is unique since $f_j^1 = \mathbf{f}(\mathbf{b}_j)$ for $j = 1, 2, \ldots, n$. So the representation of $\mathbf{F} \in X^*$ as a linear combination of $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n$ is unique and hence $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n\}$ is a basis for X^*. Therefore $\dim(X^*) = n = \dim(X)$. □

Lemma III.1.04 (continued)

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Proof (continued). Then

$$\begin{aligned}
[\mathbf{f}] &= [f_1^1, f-2^1, \ldots, f_n^1] \\
&= f_1^1[1,0,\ldots,0] + f_2^1[0,1,0,\ldots,0] + \cdots + f_n^1[0,0,\ldots,0,1] \\
&= f_1^1[\mathbf{b}^1] + f_2^1[\mathbf{b}^2] + \cdots + f_n^1[\mathbf{b}^n] \text{ since the } 1 \times n \text{ matrices in the} \\
&\qquad \text{previous line represent the functionals in } \beta \\
&= f_j^1[\mathbf{b}^j] \text{ using the Einstein summation convention} \\
&= [f_j^1 \mathbf{b}^j].
\end{aligned}$$

Now the choice of the f_j^1 for $j = 1, 2, \ldots, n$ is unique since $f_j^1 = \mathbf{f}(\mathbf{b}_j)$ for $j = 1, 2, \ldots, n$. So the representation of $\mathbf{F} \in X^*$ as a linear combination of $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n$ is unique and hence $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n\}$ is a basis for X^*. Therefore $\dim(X^*) = n = \dim(X)$. □

Lemma III.1.A

Lemma III.1.A. Given a linear functional $\mathbf{f} \in X^*$ where $X = \mathbb{R}$, there is $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{f}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle$ (the inner product on $\mathbb{R}^n$), and conversely for each $\mathbf{y} \in \mathbb{R}^n$ the mapping $\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$ is a linear functional in X^*. That is, X^* is isomorphic to $\mathbb{R}^n$ when $X = \mathbb{R}^n$.

Proof. Let $\mathbf{f} \in X^* = (\mathbb{R}^n)^*$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis of $X = \mathbb{R}^n$ and define $\mathbf{y} \in \mathbb{R}$ as $\mathbf{y} = \mathbf{f}(\mathbf{e}_i)\mathbf{e}_1 + \mathbf{f}(\mathbf{e}_2)\mathbf{e}_2 + \cdots + \mathbf{f}(\mathbf{e}_n)\mathbf{e}_n$. Then for $= x^1\mathbf{e}^1 + x^2\mathbf{e}^2 + \cdots + x^n\mathbf{e}^n)$ we have

$$
\begin{aligned}
\mathbf{f}(\mathbf{x}) &= \mathbf{f}(x^1\mathbf{e}_1 + x^2\mathbf{e}_2) + \cdots + x^n\mathbf{e}_n) \\
&= x^1\mathbf{f}(\mathbf{e}_1) + x^2\mathbf{f}(\mathbf{e}_2) + \cdots x^n\mathbf{f}(\mathbf{e}_n) \text{ since } \mathbf{f} \text{ is linear} \\
&= \langle [x^1, x^2, \ldots, x^n], [\mathbf{f}(\mathbf{e}_1), \mathbf{f}(\mathbf{e}_2), \ldots, \mathbf{f}(\mathbf{e}_n)] \rangle \\
&= \langle \mathbf{x}, \mathbf{y} \rangle,
\end{aligned}
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as claimed.

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$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \mathbf{f}(x^1\mathbf{e}_1 + x^2\mathbf{e}_2) + \cdots + x^n\mathbf{e}_n) \\ &= x^1\mathbf{f}(\mathbf{e}_1) + x^2\mathbf{f}(\mathbf{e}_2) + \cdots x^n\mathbf{f}(\mathbf{e}_n) \text{ since } \mathbf{f} \text{ is linear} \\ &= \langle [x^1, x^2, \ldots, x^n], [\mathbf{f}(\mathbf{e}_1), \mathbf{f}(\mathbf{e}_2), \ldots, \mathbf{f}(\mathbf{e}_n)] \rangle \\ &= \langle \mathbf{x}, \mathbf{y} \rangle, \end{aligned}$$

as claimed.

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Proof (continued). Conversely, for given $\mathbf{y} \in \mathbb{R}^n$, the mapping $\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$ is a functional and is linear since for scalars $a, b \in \mathbb{R}$ we have $\langle a\mathbf{x} + b\mathbf{z}, \mathbf{y} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle + b\langle \mathbf{z}, \mathbf{y} \rangle$. So the mapping $\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{y}$ is a linear functional for each $\mathbf{y} \in \mathbb{R}^n$. Therefore $X^* = (\mathbb{R}^n)^* \cong \mathbb{R}^n = X$ under the vector space isomorphism $\mathbf{f} \mapsto \langle \cdot, \mathbf{y} \rangle$ where $\mathbf{f}\mathbf{x} = \langle \mathbf{x}, \mathbf{y} \rangle$. □

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Proof. Let $\mathbf{f} \in Y^*$ where $\mathbf{f} = f_1\mathbf{b}'^1 + f_2\mathbf{b}'^2 + \cdots + f_m\mathbf{b}'^m$ where $\beta'^* = \{\mathbf{b}'^1, \mathbf{b}'^2, \ldots, \mathbf{b}'^m$ is the dual basis of Y^*, so that $\mathbf{f} = f_j\mathbf{b}'^j$ with the Einstein summation convention. Then $A^*\mathbf{f} = \mathbf{A}^*(f_j\mathbf{b}'^j) \in X^*$. Hence applying these functionals in X^* to the elements of basis β of X gives. . .

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Theorem III.1.A (continued 1)

Proof (continued).

$$
\begin{aligned}
(\mathbf{A}^*\mathbf{F})\mathbf{b}_i &= (\mathbf{A}^*(f_j\mathbf{b}_i = f_j(\mathbf{A}^*\mathbf{b}'^j)\mathbf{b}_i \text{ since scalars commute} \\
&= f_j(\mathbf{b}'^j\mathbf{A})\mathbf{b}_i \text{ by the definition of dual map } \mathbf{A}^* \text{ (here } \mathbf{f} \in Y^* \\
&\quad \text{of the definition is replaced with } \mathbf{b}'^j \in \beta'^j \subset Y^*) \\
&= f_j\mathbf{b}'^j(\mathbf{A}\mathbf{b}_i) \text{ by associativity} \\
&= f_j\mathbf{b}'^j(A(0,0,\dots,0,1,0,\dots,0)^t) \text{ since the coordinate vector} \\
&\quad \text{of } \mathbf{b}_i \text{ with respect to the ordered basis } \beta \text{ is} \\
&\quad (0,0,\dots,0,1,0,\dots,0) \text{ and matrix } A \text{ represents} \\
&\quad \mathbf{A} : X \to Y \text{ with respect to ordered bases } \beta \text{ and } \beta' \\
&= f_j\mathbf{b}'^j(a_i^1, a_i^2, \dots, a_i^m)^t \text{ since the matrix product produces the} \\
&\quad i\text{th column of } A\text{, which is a vector in } Y \text{ and this column} \\
&\quad \text{vector is in coordinates with respect to ordered basis} \\
&\quad \beta' = \{\mathbf{b}'_1, \mathbf{b}'_2, \dots, \mathbf{b}'_m\} \text{ of } Y \dots
\end{aligned}
$$

Theorem III.1.A (continued 2)

Proof (continued).

$$
\begin{aligned}
(\mathbf{A}^*\mathbf{F})\mathbf{b}_i &= f_j \mathbf{b}'^j (a_i^1, a_i^2, \ldots, a_i^m)^t \\
&= f_j a_i^j \text{ by the definition of dual basis vector } \mathbf{b}'^j : Y \to \mathbb{R} \\
&\quad \text{(or since the coordinate vector in the previous line represents} \\
&\quad a_i^k \mathbf{b}'_k \text{ and } \mathbf{b}'^j \mathbf{b}'_k = \delta_k^i \text{ by Note III.1.A).} \qquad (*)
\end{aligned}
$$

Now $\mathbf{A}^*\mathbf{f} : X \to \mathbb{R}$, so $\mathbf{A}^*\mathbf{f} \in X^*$ and can be written in terms of the dual basis $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n\}$. In doing so, say $\mathbf{A}^*\mathbf{f} = c_k \mathbf{b}^k$, we have by $(*)$ that

$$f_j a_i^j = (\mathbf{A}^*\mathbf{f})\mathbf{b}_i = (c_k \mathbf{b}^k)\mathbf{b}_i = c_k \delta_i^k \text{ (by Note III.1.A)}$$

so that $c_i = f_j a_i^j$ for $i = 1, 2, \ldots, n$.

Theorem III.1.A (continued 2)

Proof (continued).

$$\begin{aligned}(\mathbf{A}^*\mathbf{F})\mathbf{b}_i &= f_j\mathbf{b}'^j(a_i^1, a_i^2, \ldots, a_i^m)^t \\ &= f_j a_i^j \text{ by the definition of dual basis vector } \mathbf{b}'^j : Y \to \mathbb{R} \\ &\quad \text{(or since the coordinate vector in the previous line represents} \\ &\quad a_i^k\mathbf{b}'_k \text{ and } \mathbf{b}'^j\mathbf{b}'_k = \delta_k^i \text{ by Note III.1.A).} \qquad (*)\end{aligned}$$

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Theorem III.1.A (continued 2)

Proof (continued).

$$(\mathbf{A}^*\mathbf{F})\mathbf{b}_i = f_j \mathbf{b}'^j (a_i^1, a_i^2, \ldots, a_i^m)^t$$
$$= f_j a_i^j \text{ by the definition of dual basis vector } \mathbf{b}'^j : Y \to \mathbb{R}$$
$$\text{(or since the coordinate vector in the previous line represents}$$
$$a_i^k \mathbf{b}'_k \text{ and } \mathbf{b}'^j \mathbf{b}'_k = \delta_k^i \text{ by Note III.1.A).} \qquad (*)$$

Now $\mathbf{A}^*\mathbf{f} : X \to \mathbb{R}$, so $\mathbf{A}^*\mathbf{f} \in X^*$ and can be written in terms of the dual basis $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n\}$. In doing so, say $\mathbf{A}^*\mathbf{f} = c_k \mathbf{b}^k$, we have by $(*)$ that

$$f_j a_i^j = (\mathbf{A}^*\mathbf{f})\mathbf{b}_i = (c_k \mathbf{b}^k)\mathbf{b}_i = c_k \delta_i^k \text{ (by Note III.1.A)}$$

so that $c_i = f_j a_i^j$ for $i = 1, 2, \ldots, n$. Hence $\mathbf{A}^*\mathbf{f} = f_j a_i^j \mathbf{b}^i$ and the coordinate vector of $\mathbf{A}^*\mathbf{f}$ with respect to ordered basis $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n\}$ is $(f_j a_1^j, f_j a_2^j, \ldots, f_j a_n^j)$. But applying A^t to $\mathbf{f} = (f_1, f - 2, \ldots, f_m)$ (the coordinate vector of $\mathbf{f} \in Y^*$ with respect to dual basis $\beta'^* = \{\mathbf{b}'^1, \mathbf{b}'^2, \ldots, \mathbf{b}'^m\}$) gives...

Theorem III.1.A (continued 3)

Proof (continued).

$$\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{pmatrix}^T \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^m \\ a_2^1 & a_2^2 & \cdots & a_2^m \\ \vdots & \vdots & \ddots & \vdots \\ a_n^1 & a_n^2 & \cdots & a_n^m \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$$

$$= \begin{pmatrix} a_1^1 f_1 + a_1^2 f_2 + \cdots + a_1^m f_m \\ a_2^1 f_1 + a_2^2 f_2 + \cdots + a_2^m f_m \\ \vdots \\ a_n^1 f_1 + a_n^2 f_2 + \cdots + a_n^m f_m \end{pmatrix} = \begin{pmatrix} f_j a_1^j \\ f_j a_2^j \\ \vdots \\ f_j a_n^j \end{pmatrix}.$$

Therefore, the $n \times m$ matrix $A^t = \left([\mathbf{A}]_\beta^{\beta'}\right)^t$ has the same effect as the dual map $\mathbf{A}^* : Y^* \to X^*$ with respect to the ordered bases β'^* and β^*. That is, $[\mathbf{A}^*]_{\beta'^*}^{\beta^*} = \left([\mathbf{A}]_\beta^{\beta'}\right)^t$, as claimed. □

Theorem III.1.A (continued 3)

Proof (continued).

$$\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{pmatrix}^T \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^m \\ a_2^1 & a_2^2 & \cdots & a_2^m \\ \vdots & \vdots & \ddots & \vdots \\ a_n^1 & a_n^2 & \cdots & a_n^m \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$$

$$= \begin{pmatrix} a_1^1 f_1 + a_1^2 f_2 + \cdots + a_1^m f_m \\ a_2^1 f_1 + a_2^2 f_2 + \cdots + a_2^m f_m \\ \vdots \\ a_n^1 f_1 + a_n^2 f_2 + \cdots + a_n^m f_m \end{pmatrix} = \begin{pmatrix} f_j a_1^j \\ f_j a_2^j \\ \vdots \\ f_j a_n^j \end{pmatrix}.$$

Therefore, the $n \times m$ matrix $A^t = \left([\mathbf{A}]_\beta^{\beta'} \right)^t$ has the same effect as the dual map $\mathbf{A}^* : Y^* \to X^*$ with respect to the ordered bases β'^* and β^*. That is, $[\mathbf{A}^*]_{\beta'^*}^{\beta^*} = \left([\mathbf{A}]_\beta^{\beta'} \right)^t$, as claimed. □

Theorem III.1.B

Theorem III.1.B. Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\beta' = \{\mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_n\}$ be bases for X and let $\beta^* = \{\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n\}$ be the dual basis of β (so that β^* is a basis of X^*). With $\beta'^* = \{\mathbf{b}'^1, \mathbf{b}'^2, \ldots, \mathbf{b}'^n\}$ the dual basis of β', for $\mathbf{f} \in X^*$ where

$$\mathbf{f} = \sum_{i=1}^{n} f_i \mathbf{b}^i = f_i \mathbf{b}^i = \sum_{i=1}^{n} f'_i \mathbf{b}'^i = f'_i \mathbf{b}'^i$$

we have $f'_i = b^j_i f_j$ where the b^j_i are coordinates of $\mathbf{b}'_i \in \beta'$ with respect to ordered basis β (that is, b^j_i satisfies $\mathbf{b}'_i = \sum_{j=1}^{n} b^j_i \mathbf{b}_j = b^j_i \mathbf{b}_j$).

Proof. The matrix which converts coordinate vectors with respect to ordered basis β to coordinate vectors with respect to ordered basis β' is ...

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$$\mathbf{f} = \sum_{i=1}^{n} f_i \mathbf{b}^i = f_i \mathbf{b}^i = \sum_{i=1}^{n} f'_i \mathbf{b}'^i = f'_i \mathbf{b}'^i$$

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Proof. The matrix which converts coordinate vectors with respect to ordered basis β to coordinate vectors with respect to ordered basis β' is ...

Theorem III.1.B (continued 1)

Proof (continued).

$$[\mathbf{I}]_{\beta}^{\beta'} = \left([\mathbf{I}]_{\beta'}^{\beta}\right)^{-1} = \begin{bmatrix} b_1^1 & b_2^1 & \cdots & b_n^1 \\ b_1^2 & b_2^2 & \cdots & b_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1^n & b_2^n & \cdots & b_n^n \end{bmatrix}^{-1}$$

since $\mathbf{b}_i' = b_i^j \mathbf{b}_j$ by the previous Note (here β and β' are interchanged from the Note). By Theorem III.1.A, the dual map of $\mathbf{I} : X \to X$ (here $\mathbf{I} \in L(X, X)$ maps each vector to itself but matrix $[\mathbf{I}]_{\beta}^{\beta'}$ allows us to represent this map as a conversion of coordinate vectors with respect to β to coordinate vectors with respect to β'), $\mathbf{I}^* : X^* \to X^*$ (satisfying $\mathbf{I}^*(\mathbf{f}) = \mathbf{f} \circ \mathbf{I}$ for all $\mathbf{F} \in X^*$, by the definition of dual map), has matrix representation...

Theorem III.1.B (continued 1)

Proof (continued).

$$[\mathbf{I}]_{\beta}^{\beta'} = \left([\mathbf{I}]_{\beta'}^{\beta}\right)^{-1} = \begin{bmatrix} b_1^1 & b_2^1 & \cdots & b_n^1 \\ b_1^2 & b_2^2 & \cdots & b_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1^n & b_2^n & \cdots & b_n^n \end{bmatrix}^{-1}$$

since $\mathbf{b}'_i = b_i^j \mathbf{b}_j$ by the previous Note (here β and β' are interchanged from the Note). By Theorem III.1.A, the dual map of $\mathbf{I} : X \to X$ (here $\mathbf{I} \in L(X, X)$ maps each vector to itself but matrix $[\mathbf{I}]_{\beta}^{\beta'}$ allows us to represent this map as a conversion of coordinate vectors with respect to β to coordinate vectors with respect to β'), $\mathbf{I}^* : X^* \to X^*$ (satisfying $\mathbf{I}^*(\mathbf{f}) = \mathbf{f} \circ \mathbf{I}$ for all $\mathbf{F} \in X^*$, by the definition of dual map), has matrix representation...

Theorem III.1.B (continued 2)

Proof (continued).

$$[\mathbf{I}^*]_{\beta^*}^{\beta'^*} = \left([\mathbf{I}]_{\beta'}^{\beta}\right)^T = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ b_2^1 & b_2^2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & \cdots & b_n^n \end{bmatrix}.$$

So for $\mathbf{f} \in X^*$ where

$$\mathbf{f} = \sum_{i=1}^{n} f_i \mathbf{b}^i = f_i \mathbf{b}^i = \sum_{i=1}^{n} f_i' \mathbf{b}'^i = f_i \mathbf{b}'^i$$

(so that the coordinate [row] vectors of $\mathbf{f}$ are $(f_1, f_2, \ldots, f_n)$ and $(f_1', f_2', \ldots, f_n')$ with respect to ordered bases β^* and β'^*, respectively), we have...

Theorem III.1.B (continued 3)

Proof (continued).

$$\begin{bmatrix} f_1' \\ f_2' \\ \vdots \\ f_n' \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ b_2^1 & b_2^2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} b_1^j f_j \\ b_2^j f_j \\ \vdots \\ b_n^j f_j \end{bmatrix},$$

so that $f_i' = b_i^j f_j$, as claimed. □