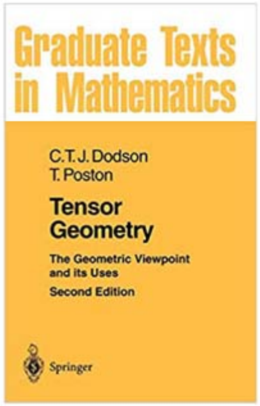
Differential Geometry

Chapter IV. Metric Vector Spaces

IV.1. Metrics—Proofs of Theorems



Lemma IV.1.A

Lemma IV.1.A. For $\|\cdot\|$ a norm on vector space X, we have $\|\mathbf{0}\| = 0$ and for all $\mathbf{x} \in X$ that $\|\mathbf{x}\| \geq 0$.

Proof. By (N ii) we have for $a = 2$ that $\|\mathbf{0}\| = \|\mathbf{0}2\| = |2|\|\mathbf{0}\| = 2\|\mathbf{0}\|$ and so $\|\mathbf{0}\| = 0$.

ASSUME that for some $\mathbf{x} \in X$ we have $\|\mathbf{x}\| < 0$. Then by (N ii) with $a = -1$ we have

$$\|-\mathbf{x}\| = \|\mathbf{x}(-1)\| = |-1|\|\mathbf{x}\| = \|\mathbf{x}\| < 0.$$

Then by the Triangle Inequality (N iii),

$$\|\mathbf{0}\| = \|\mathbf{x} - \mathbf{x}\| = \|\mathbf{x} + \mathbf{x}(-1)\| \leq \|\mathbf{x}\| + \|\mathbf{x}(-1)\| = 2\|\mathbf{x}\| < 0.$$

But $\|\mathbf{0}\| = 0$ as shown above, so this is a CONTRADICTION. Hence the assumption is false and we have $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in X$, as claimed. □

Lemma IV.1.07. Schwarz's Inequality

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In any inner product space $(X, \mathbf{G})$ (with positive definite $\mathbf{G}$) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$ with equality for nonzero $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{y} = \mathbf{x}a$ for some $a \in \mathbb{R}$, and if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$ then equality holds.

Proof. For any $a \in \mathbb{R}$ we have by linearity and symmetry of an inner product

$$(\mathbf{x}a - \mathbf{y}, \mathbf{x}a - \mathbf{y}) = \mathbf{x}a \cdot \mathbf{x}a - \mathbf{y} \cdot \mathbf{x}a - \mathbf{x}a \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = (\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}).$$

Since $\mathbf{G}$ is positive definite, then $(\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}) \geq 0$ for $a \in \mathbb{R}$. Therefore the quadratic equation $(\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}) = 0$ cannot have distinct real roots (for the quadratic would then be negative between the two real roots), hence the discriminant must be nonpositive: $(2\mathbf{x} \cdot \mathbf{y})^2 - 4(\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y}) \leq 0$. This implies $(\mathbf{x} \cdot \mathbf{y})^2 \leq (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$ and $|(\mathbf{x} \cdot \mathbf{y})| = \sqrt{(\mathbf{x} \cdot \mathbf{y})^2} \leq \sqrt{\mathbf{x} \cdot \mathbf{x}}\sqrt{\mathbf{y} \cdot \mathbf{y}}$, so $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$, as claimed.

Lemma IV.1.07 (continued)

Lemma IV.1.07. Schwarz's Inequality.
In any inner product space $(X, \mathbf{G})$ (with positive definite $\mathbf{G}$) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$ with equality for nonzero $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{y} = \mathbf{x}a$ for some $a \in \mathbb{R}$, and if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$ then equality holds.

Proof (continued). Next, for $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}$, the quadratic equation has one real root (namely, $a = \dfrac{1(\mathbf{x} \cdot \mathbf{y})}{2(\mathbf{x} \cdot \mathbf{x})} = \dfrac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|^2}$) and for this value of a we have $(\mathbf{x}a - \mathbf{y}) \cdot (\mathbf{x}a = \mathbf{y}) = 0$. Since the inner product is positive definite (or negative definite, say) then $\mathbf{x}a - \mathbf{y} = \mathbf{0}$ or $\mathbf{y} = \mathbf{x}a$. Finally, if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$, then

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{x}a) = a(\mathbf{x} \cdot \mathbf{x}) = a\|\mathbf{x}\|^2 = \|\mathbf{x}\| = \mathbf{x}a\| = \|\mathbf{x}\|\|\mathbf{y}\|.$$

□

Theorem IV.1.09

Theorem IV.1.09. For any non-degenerate bilinear form $\mathbf{F}$ on a vector space X, the map $\mathbf{F}_\downarrow : X \to X^*$ defined as $\mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y})$, is linear and an isomorphism.

Proof. For any $\mathbf{x}, \mathbf{x}', \mathbf{y} \in X$ and $a \in \mathbb{R}$,

$$\begin{aligned} \mathbf{F}_\downarrow(\mathbf{x}+\mathbf{x}'))(\mathbf{Y}) &= (\mathbf{x}+\mathbf{x}')^*(\mathbf{y}) = \mathbf{F}(\mathbf{x}+\mathbf{x}', \mathbf{y}) \\ &= \mathbf{F}(\mathbf{x}, \mathbf{y}) + \mathbf{F}(\mathbf{x}', \mathbf{y}) \text{ since } \mathbf{F} \text{ is bilinear} \\ &= \mathbf{x}^*(\mathbf{y}) + (\mathbf{x}')^*(\mathbf{y}) = \mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) + \mathbf{F}_\downarrow(\mathbf{x}')(\mathbf{y}) \\ &= (\mathbf{F}_\downarrow(\mathbf{x}) + \mathbf{F}_\downarrow(\mathbf{x}'))(\mathbf{y}); \end{aligned}$$

that is $\mathbf{F}_\downarrow(\mathbf{x}+\mathbf{x}') = \mathbf{F}_\downarrow(\mathbf{x} + \mathbf{F}_\downarrow(\mathbf{x}')$. Also,

$$\begin{aligned} \mathbf{F}(\mathbf{x}a)(\mathbf{y}) &= (\mathbf{x}a)^*(\mathbf{y}) = (\mathbf{x}a)\cdot\mathbf{y} = \mathbf{F}(\mathbf{x}a, \mathbf{y}) \\ &= a\mathbf{F}(\mathbf{x}, \mathbf{y}) \text{ since } \mathbf{F} \text{ is bilinear} \\ &= a(\mathbf{x}\cdot\mathbf{y}) = a(\mathbf{x}^*(\mathbf{y})) = a(\mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y})) = (a\mathbf{F}_\downarrow(\mathbf{x}))(\mathbf{y}); \end{aligned}$$

that is $\mathbf{F}_\downarrow(\mathbf{X}a) = (\mathbf{F}_\downarrow(\mathbf{x}))a$. Hence $\mathbf{F}_\downarrow$ is linear.

Theorem IV.1.09 (continued)

Proof (continued). Suppose $\mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) = 0$ for all $\mathbf{y} \in X$; that is, $\mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{0}$. Then for all $\mathbf{y} \in X$, $\mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) = \mathbf{x}^*(\mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}) = 0$ and since $\mathbf{F}$ is non-degenerate by hypothesis then $\mathbf{x} = \mathbf{0}$. So $\ker(\mathbf{F}_\downarrow) = \{\mathbf{x} \in X \mid \mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{0}\}$. So $\dim(\ker(\mathbf{F}_\downarrow)) = \text{nullity}(\mathbf{F}_\downarrow) = 0$ and by the Rank-Nullity Equation (see "Theorem 2.5. Rank Equation" in my online note for Linear Algebra [MATH 2010] on "2.2. The Rank of a Matrix," or Theorem I.2.10 of the Dodson and Poston) we have $\dim(\text{Im}(\mathbf{F}_\downarrow)) = \dim(\ker(\mathbf{F}_\downarrow)) + \dim(\text{Im}(\mathbf{F}_\downarrow))$ where $\dim(\text{Im}(\mathbf{F}_\downarrow)) = \dim(\mathbf{F}_\downarrow X)$. So $\dim(\mathbf{F}_\downarrow X) = \dim(X) = \dim(X^*)$ (by Lemma III.1.04) and hence $\mathbf{F}_\downarrow X = X^*$ (since X and X^* are finite dimensional). That is, $\mathbf{F}_\downarrow$ is onto (surjective), one to one (injective; since $\dim(\ker(\mathbf{F}_\downarrow)) = 0$), and linear. That is, $\mathbf{F}_\downarrow$ is an isomorphism, as claimed. □

Lemma IV.1.11

Lemma IV.1.11. A non-degenerate bilinear form $\mathbf{F}$ on a vector space X induces a bilinear form $\mathbf{F}^*$ on X^* by

$$\mathbf{F}^*(\mathbf{f}, \mathbf{g}) = \mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{F}_\uparrow(\mathbf{g}))$$

which is non-degenerate. In addition, if $\mathbf{F}$ is symmetric/anti-symmetric/positive definite/negative definite/indefinite then so is $\mathbf{F}^*$.

Proof. Since $\mathbf{F}$ is bilinear (i.e., linear in both positions) then $\mathbf{F}^*$ is bilinear.

Suppose $\mathbf{F}$ is non-degenerate. Let $\mathbf{f} \in X^*$ be such that $\mathbf{F}^*(\mathbf{F}, \mathbf{g}) = 0$ for all $\mathbf{g} \in X^*$. Then $\mathbf{F}^*(\mathbf{F}, \mathbf{g}) = \mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{F}_\uparrow(\mathbf{g})) = 0$ for all $\mathbf{g} \in X^*$. Since $\mathbf{F}_\uparrow : X^* \to X$ is onto, then we can equivalently say $\mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{y}) = 0$ for all $\mathbf{y} \in X$ (we take $\mathbf{y} = \mathbf{F}_\uparrow(\mathbf{g})$). Since $\mathbf{F}$ is non-degenerate, this implies $\mathbf{F}_\uparrow(\mathbf{f}) = 0$ and since $\mathbf{F}_\uparrow$ is linear and one to one, this implies that $\mathbf{f} = \mathbf{0}$. Therefore $\mathbf{F}_\uparrow$ is non-degenerate, as claimed.

The claims about symmetry/etc. are to be shown in Exercise IV.1.8. □