Differential Geometry

Chapter IV. Metric Vector Spaces

IV.1. Metrics—Proofs of Theorems

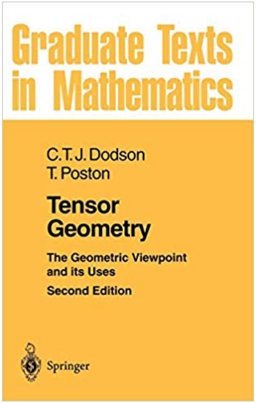

Table of contents

1. Lemma IV.1.A
2. Lemma IV.1.07. Schwarz's Inequality
3. Theorem IV.1.09
4. Lemma IV.1.11

Lemma IV.1.A

Lemma IV.1.A. For $\|\cdot\|$ a norm on vector space X, we have $\|\mathbf{0}\| = 0$ and for all $\mathbf{x} \in X$ that $\|\mathbf{x}\| \geq 0$.

Proof. By (N ii) we have for $a = 2$ that $\|\mathbf{0}\| = \|\mathbf{0}2\| = |2|\|\mathbf{0}\| = 2\|\mathbf{0}\|$ and so $\|\mathbf{0}\| = 0$.

Lemma IV.1.A

Lemma IV.1.A. For $\|\cdot\|$ a norm on vector space X, we have $\|\mathbf{0}\| = 0$ and for all $\mathbf{x} \in X$ that $\|\mathbf{x}\| \geq 0$.

Proof. By (N ii) we have for $a = 2$ that $\|\mathbf{0}\| = \|\mathbf{0}2\| = |2|\|\mathbf{0}\| = 2\|\mathbf{0}\|$ and so $\|\mathbf{0}\| = 0$.

ASSUME that for some $\mathbf{x} \in X$ we have $\|\mathbf{x}\| < 0$. Then by (N ii) with $a = -1$ we have

$$\|-\mathbf{x}\| = \|\mathbf{x}(-1)\| = |-1|\|\mathbf{x}\| = \|\mathbf{x}\| < 0.$$

Then by the Triangle Inequality (N iii),

$$\|\mathbf{0}\| = \|\mathbf{x} - \mathbf{x}\| = \|\mathbf{x} + \mathbf{x}(-1)\| \leq \|\mathbf{x}\| + \|\mathbf{x}(-1)\| = 2\|\mathbf{x}\| < 0.$$

But $\|\mathbf{0}\| = 0$ as shown above, so this is a CONTRADICTION. Hence the assumption is false and we have $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in X$, as claimed. □

Lemma IV.1.A

Lemma IV.1.A. For $\|\cdot\|$ a norm on vector space X, we have $\|\mathbf{0}\| = 0$ and for all $\mathbf{x} \in X$ that $\|\mathbf{x}\| \geq 0$.

Proof. By (N ii) we have for $a = 2$ that $\|\mathbf{0}\| = \|\mathbf{0}2\| = |2|\|\mathbf{0}\| = 2\|\mathbf{0}\|$ and so $\|\mathbf{0}\| = 0$.

ASSUME that for some $\mathbf{x} \in X$ we have $\|\mathbf{x}\| < 0$. Then by (N ii) with $a = -1$ we have

$$\|-\mathbf{x}\| = \|\mathbf{x}(-1)\| = |-1|\|\mathbf{x}\| = \|\mathbf{x}\| < 0.$$

Then by the Triangle Inequality (N iii),

$$\|\mathbf{0}\| = \|\mathbf{x} - \mathbf{x}\| = \|\mathbf{x} + \mathbf{x}(-1)\| \leq \|\mathbf{x}\| + \|\mathbf{x}(-1)\| = 2\|\mathbf{x}\| < 0.$$

But $\|\mathbf{0}\| = 0$ as shown above, so this is a CONTRADICTION. Hence the assumption is false and we have $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in X$, as claimed. □

Lemma IV.1.07. Schwarz's Inequality

Lemma IV.1.07. Schwarz's Inequality.
In any inner product space $(X, \mathbf{G})$ (with positive definite $\mathbf{G}$) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$ with equality for nonzero $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{y} = \mathbf{x}a$ for some $a \in \mathbb{R}$, and if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$ then equality holds.

Proof. For any $a \in \mathbb{R}$ we have by linearity and symmetry of an inner product

$$(\mathbf{x}a - \mathbf{y}, \mathbf{x}a - \mathbf{y}) = \mathbf{x}a \cdot \mathbf{x}a - \mathbf{y} \cdot \mathbf{x}a - \mathbf{x}a \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = (\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}).$$

Since $\mathbf{G}$ is positive definite, then $(\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}) \geq 0$ for $a \in \mathbb{R}$.

Lemma IV.1.07. Schwarz's Inequality

Lemma IV.1.07. Schwarz's Inequality.
In any inner product space $(X, \mathbf{G})$ (with positive definite $\mathbf{G}$) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$ with equality for nonzero $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{y} = \mathbf{x}a$ for some $a \in \mathbb{R}$, and if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$ then equality holds.

Proof. For any $a \in \mathbb{R}$ we have by linearity and symmetry of an inner product

$$(\mathbf{x}a - \mathbf{y}, \mathbf{x}a - \mathbf{y}) = \mathbf{x}a \cdot \mathbf{x}a - \mathbf{y} \cdot \mathbf{x}a - \mathbf{x}a \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = (\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}).$$

Since $\mathbf{G}$ is positive definite, then $(\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}) \geq 0$ for $a \in \mathbb{R}$. Therefore the quadratic equation $(\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}) = 0$ cannot have distinct real roots (for the quadratic would then be negative between the two real roots), hence the discriminant must be nonpositive: $(2\mathbf{x} \cdot \mathbf{y})^2 - 4(\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y}) \leq 0$. This implies $(\mathbf{x} \cdot \mathbf{y})^2 \leq (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$ and $|(\mathbf{x} \cdot \mathbf{y})| = \sqrt{(\mathbf{x} \cdot \mathbf{y})^2} \leq \sqrt{\mathbf{x} \cdot \mathbf{x}}\sqrt{\mathbf{y} \cdot \mathbf{y}}$, so $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$, as claimed.

Lemma IV.1.07. Schwarz's Inequality

Lemma IV.1.07. Schwarz's Inequality.
In any inner product space $(X, \mathbf{G})$ (with positive definite $\mathbf{G}$) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$ with equality for nonzero $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{y} = \mathbf{x}a$ for some $a \in \mathbb{R}$, and if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$ then equality holds.

Proof. For any $a \in \mathbb{R}$ we have by linearity and symmetry of an inner product

$$(\mathbf{x}a - \mathbf{y}, \mathbf{x}a - \mathbf{y}) = \mathbf{x}a \cdot \mathbf{x}a - \mathbf{y} \cdot \mathbf{x}a - \mathbf{x}a \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = (\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}).$$

Since $\mathbf{G}$ is positive definite, then $(\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}) \geq 0$ for $a \in \mathbb{R}$. Therefore the quadratic equation $(\mathbf{x} \cdot \mathbf{x})a^2 - (2\mathbf{x} \cdot \mathbf{y})a + (\mathbf{y} \cdot \mathbf{y}) = 0$ cannot have distinct real roots (for the quadratic would then be negative between the two real roots), hence the discriminant must be nonpositive: $(2\mathbf{x} \cdot \mathbf{y})^2 - 4(\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y}) \leq 0$. This implies $(\mathbf{x} \cdot \mathbf{y})^2 \leq (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$ and $|(\mathbf{x} \cdot \mathbf{y})| = \sqrt{(\mathbf{x} \cdot \mathbf{y})^2} \leq \sqrt{\mathbf{x} \cdot \mathbf{x}}\sqrt{\mathbf{y} \cdot \mathbf{y}}$, so $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$, as claimed.

Lemma IV.1.07 (continued)

Lemma IV.1.07. Schwarz's Inequality.
In any inner product space $(X, \mathbf{G})$ (with positive definite $\mathbf{G}$) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$ with equality for nonzero $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{y} = \mathbf{x}a$ for some $a \in \mathbb{R}$, and if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$ then equality holds.

Proof (continued). Next, for $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}$, the quadratic equation has one real root (namely, $a = \dfrac{1(\mathbf{x} \cdot \mathbf{y})}{2(\mathbf{x} \cdot \mathbf{x})} = \dfrac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|^2}$) and for this value of a we have $(\mathbf{x}a - \mathbf{y}) \cdot (\mathbf{x}a = \mathbf{y}) = 0$. Since the inner product is positive definite (or negative definite, say) then $\mathbf{x}a - \mathbf{y} = \mathbf{0}$ or $\mathbf{y} = \mathbf{x}a$.

Lemma IV.1.07 (continued)

Lemma IV.1.07. Schwarz's Inequality.
In any inner product space $(X, \mathbf{G})$ (with positive definite $\mathbf{G}$) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$ with equality for nonzero $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{y} = \mathbf{x}a$ for some $a \in \mathbb{R}$, and if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$ then equality holds.

Proof (continued). Next, for $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}$, the quadratic equation has one real root (namely, $a = \dfrac{1(\mathbf{x} \cdot \mathbf{y})}{2(\mathbf{x} \cdot \mathbf{x})} = \dfrac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|^2}$) and for this value of a we have $(\mathbf{x}a - \mathbf{y}) \cdot (\mathbf{x}a = \mathbf{y}) = 0$. Since the inner product is positive definite (or negative definite, say) then $\mathbf{x}a - \mathbf{y} = \mathbf{0}$ or $\mathbf{y} = \mathbf{x}a$. Finally, if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$, then

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{x}a) = a(\mathbf{x} \cdot \mathbf{x}) = a\|\mathbf{x}\|^2 = \|\mathbf{x}\| = \mathbf{x}a\| = \|\mathbf{x}\|\|\mathbf{y}\|.$$

□

Lemma IV.1.07 (continued)

Lemma IV.1.07. Schwarz's Inequality.
In any inner product space $(X, \mathbf{G})$ (with positive definite $\mathbf{G}$) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$ with equality for nonzero $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{y} = \mathbf{x}a$ for some $a \in \mathbb{R}$, and if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$ then equality holds.

Proof (continued). Next, for $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}$, the quadratic equation has one real root (namely, $a = \dfrac{1(\mathbf{x} \cdot \mathbf{y})}{2(\mathbf{x} \cdot \mathbf{x})} = \dfrac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|^2}$) and for this value of a we have $(\mathbf{x}a - \mathbf{y}) \cdot (\mathbf{x}a = \mathbf{y}) = 0$. Since the inner product is positive definite (or negative definite, say) then $\mathbf{x}a - \mathbf{y} = \mathbf{0}$ or $\mathbf{y} = \mathbf{x}a$. Finally, if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$, then

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{x}a) = a(\mathbf{x} \cdot \mathbf{x}) = a\|\mathbf{x}\|^2 = \|\mathbf{x}\| = \mathbf{x}a\| = \|\mathbf{x}\|\|\mathbf{y}\|.$$

□

Theorem IV.1.09

Theorem IV.1.09. For any non-degenerate bilinear form $\mathbf{F}$ on a vector space X, the map $\mathbf{F}_\downarrow : X \to X^*$ defined as $\mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y})$, is linear and an isomorphism.

Proof. For any $\mathbf{x}, \mathbf{x}', \mathbf{y} \in X$ and $a \in \mathbb{R}$,

$$\begin{aligned} \mathbf{F}_\downarrow(\mathbf{x}+\mathbf{x}'))(\mathbf{Y}) &= (\mathbf{x}+\mathbf{x}')^*(\mathbf{y}) = \mathbf{F}(\mathbf{x}+\mathbf{x}', \mathbf{y}) \\ &= \mathbf{F}(\mathbf{x}, \mathbf{y}) + \mathbf{F}(\mathbf{x}', \mathbf{y}) \text{ since } \mathbf{F} \text{ is bilinear} \\ &= \mathbf{x}^*(\mathbf{y}) + (\mathbf{x}')^*(\mathbf{y}) = \mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) + \mathbf{F}_\downarrow(\mathbf{x}')(\mathbf{y}) \\ &= (\mathbf{F}_\downarrow(\mathbf{x}) + \mathbf{F}_\downarrow(\mathbf{x}'))(\mathbf{y}); \end{aligned}$$

that is $\mathbf{F}_\downarrow(\mathbf{x}+\mathbf{x}') = \mathbf{F}_\downarrow(\mathbf{x} + \mathbf{F}_\downarrow(\mathbf{x}')$.

Theorem IV.1.09

Theorem IV.1.09. For any non-degenerate bilinear form $\mathbf{F}$ on a vector space X, the map $\mathbf{F}_\downarrow : X \to X^*$ defined as $\mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y})$, is linear and an isomorphism.

Proof. For any $\mathbf{x}, \mathbf{x}', \mathbf{y} \in X$ and $a \in \mathbb{R}$,

$$\begin{aligned} \mathbf{F}_\downarrow(\mathbf{x} + \mathbf{x}'))(\mathbf{Y}) &= (\mathbf{x} + \mathbf{x}')^*(\mathbf{y}) = \mathbf{F}(\mathbf{x} + \mathbf{x}', \mathbf{y}) \\ &= \mathbf{F}(\mathbf{x}, \mathbf{y}) + \mathbf{F}(\mathbf{x}', \mathbf{y}) \text{ since } \mathbf{F} \text{ is bilinear} \\ &= \mathbf{x}^*(\mathbf{y}) + (\mathbf{x}')^*(\mathbf{y}) = \mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) + \mathbf{F}_\downarrow(\mathbf{x}')(\mathbf{y}) \\ &= (\mathbf{F}_\downarrow(\mathbf{x}) + \mathbf{F}_\downarrow(\mathbf{x}'))(\mathbf{y}); \end{aligned}$$

that is $\mathbf{F}_\downarrow(\mathbf{x} + \mathbf{x}') = \mathbf{F}_\downarrow(\mathbf{x} + \mathbf{F}_\downarrow(\mathbf{x}')$. Also,

$$\begin{aligned} \mathbf{F}(\mathbf{x}a)(\mathbf{y}) &= (\mathbf{x}a)^*(\mathbf{y}) = (\mathbf{x}a) \cdot \mathbf{y} = \mathbf{F}(\mathbf{x}a, \mathbf{y}) \\ &= a\mathbf{F}(\mathbf{x}, \mathbf{y}) \text{ since } \mathbf{F} \text{ is bilinear} \\ &= a(\mathbf{x} \cdot \mathbf{y}) = a(\mathbf{x}^*(\mathbf{y})) = a(\mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y})) = (a\mathbf{F}_\downarrow(\mathbf{x}))(\mathbf{y}); \end{aligned}$$

that is $\mathbf{F}_\downarrow(\mathbf{X}a) = (\mathbf{F}_\downarrow(\mathbf{x}))a$. Hence $\mathbf{F}_\downarrow$ is linear.

Theorem IV.1.09

Theorem IV.1.09. For any non-degenerate bilinear form $\mathbf{F}$ on a vector space X, the map $\mathbf{F}_\downarrow : X \to X^*$ defined as $\mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y})$, is linear and an isomorphism.

Proof. For any $\mathbf{x}, \mathbf{x}', \mathbf{y} \in X$ and $a \in \mathbb{R}$,

$$\begin{aligned} \mathbf{F}_\downarrow(\mathbf{x} + \mathbf{x}'))(\mathbf{Y}) &= (\mathbf{x} + \mathbf{x}')^*(\mathbf{y}) = \mathbf{F}(\mathbf{x} + \mathbf{x}', \mathbf{y}) \\ &= \mathbf{F}(\mathbf{x}, \mathbf{y}) + \mathbf{F}(\mathbf{x}', \mathbf{y}) \text{ since } \mathbf{F} \text{ is bilinear} \\ &= \mathbf{x}^*(\mathbf{y}) + (\mathbf{x}')^*(\mathbf{y}) = \mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) + \mathbf{F}_\downarrow(\mathbf{x}')(\mathbf{y}) \\ &= (\mathbf{F}_\downarrow(\mathbf{x}) + \mathbf{F}_\downarrow(\mathbf{x}'))(\mathbf{y}); \end{aligned}$$

that is $\mathbf{F}_\downarrow(\mathbf{x} + \mathbf{x}') = \mathbf{F}_\downarrow(\mathbf{x} + \mathbf{F}_\downarrow(\mathbf{x}')$. Also,

$$\begin{aligned} \mathbf{F}(\mathbf{x}a)(\mathbf{y}) &= (\mathbf{x}a)^*(\mathbf{y}) = (\mathbf{x}a) \cdot \mathbf{y} = \mathbf{F}(\mathbf{x}a, \mathbf{y}) \\ &= a\mathbf{F}(\mathbf{x}, \mathbf{y}) \text{ since } \mathbf{F} \text{ is bilinear} \\ &= a(\mathbf{x} \cdot \mathbf{y}) = a(\mathbf{x}^*(\mathbf{y})) = a(\mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y})) = (a\mathbf{F}_\downarrow(\mathbf{x}))(\mathbf{y}); \end{aligned}$$

that is $\mathbf{F}_\downarrow(\mathbf{X}a) = (\mathbf{F}_\downarrow(\mathbf{x}))a$. Hence $\mathbf{F}_\downarrow$ is linear.

Theorem IV.1.09 (continued)

Proof (continued). Suppose $\mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) = 0$ for all $\mathbf{y} \in X$; that is, $\mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{0}$. Then for all $\mathbf{y} \in X$, $\mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) = \mathbf{x}^*(\mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}) = 0$ and since $\mathbf{F}$ is non-degenerate by hypothesis then $\mathbf{x} = \mathbf{0}$. So $\ker(\mathbf{F}_\downarrow) = \{\mathbf{x} \in X \mid \mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{0}\}$. So $\dim(\ker(\mathbf{F}_\downarrow)) = \text{nullity}(\mathbf{F}_\downarrow) = 0$ and by the Rank-Nullity Equation (see "Theorem 2.5. Rank Equation" in my online note for Linear Algebra [MATH 2010] on "2.2. The Rank of a Matrix," or Theorem I.2.10 of the Dodson and Poston) we have $\dim(\text{Im}(\mathbf{F}_\downarrow)) = \dim(\ker(\mathbf{F}_\downarrow)) + \dim(\text{Im}(\mathbf{F}_\downarrow))$ where $\dim(\text{Im}(\mathbf{F}_\downarrow)) = \dim(\mathbf{F}_\downarrow X)$. So $\dim(\mathbf{F}_\downarrow X) = \dim(X) = \dim(X^*)$ (by Lemma III.1.04) and hence $\mathbf{F}_\downarrow X = X^*$ (since X and X^* are finite dimensional). That is, $\mathbf{F}_\downarrow$ is onto (surjective), one to one (injective; since $\dim(\ker(\mathbf{F}_\downarrow)) = 0$), and linear. That is, $\mathbf{F}_\downarrow$ is an isomorphism, as claimed. □

Theorem IV.1.09 (continued)

Proof (continued). Suppose $\mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) = 0$ for all $\mathbf{y} \in X$; that is, $\mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{0}$. Then for all $\mathbf{y} \in X$, $\mathbf{F}_\downarrow(\mathbf{x})(\mathbf{y}) = \mathbf{x}^*(\mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}) = 0$ and since $\mathbf{F}$ is non-degenerate by hypothesis then $\mathbf{x} = \mathbf{0}$. So $\ker(\mathbf{F}_\downarrow) = \{\mathbf{x} \in X \mid \mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{0}\}$. So $\dim(\ker(\mathbf{F}_\downarrow)) = \text{nullity}(\mathbf{F}_\downarrow) = 0$ and by the Rank-Nullity Equation (see "Theorem 2.5. Rank Equation" in my online note for Linear Algebra [MATH 2010] on "2.2. The Rank of a Matrix," or Theorem I.2.10 of the Dodson and Poston) we have $\dim(\text{Im}(\mathbf{F}_\downarrow)) = \dim(\ker(\mathbf{F}_\downarrow)) + \dim(\text{Im}(\mathbf{F}_\downarrow))$ where $\dim(\text{Im}(\mathbf{F}_\downarrow)) = \dim(\mathbf{F}_\downarrow X)$. So $\dim(\mathbf{F}_\downarrow X) = \dim(X) = \dim(X^*)$ (by Lemma III.1.04) and hence $\mathbf{F}_\downarrow X = X^*$ (since X and X^* are finite dimensional). That is, $\mathbf{F}_\downarrow$ is onto (surjective), one to one (injective; since $\dim(\ker(\mathbf{F}_\downarrow)) = 0$), and linear. That is, $\mathbf{F}_\downarrow$ is an isomorphism, as claimed. □

Lemma IV.1.11

Lemma IV.1.11. A non-degenerate bilinear form $\mathbf{F}$ on a vector space X induces a bilinear form $\mathbf{F}^*$ on X^* by

$$\mathbf{F}^*(\mathbf{f},\mathbf{g}) = \mathbf{F}(\mathbf{F}_{\uparrow}(\mathbf{f}),\mathbf{F}_{\uparrow}(\mathbf{g}))$$

which is non-degenerate. In addition, if $\mathbf{F}$ is symmetric/anti-symmetric/positive definite/negative definite/indefinite then so is $\mathbf{F}^*$.

Proof. Since $\mathbf{F}$ is bilinear (i.e., linear in both positions) then $\mathbf{F}^*$ is bilinear.

Lemma IV.1.11

Lemma IV.1.11. A non-degenerate bilinear form $\mathbf{F}$ on a vector space X induces a bilinear form $\mathbf{F}^*$ on X^* by

$$\mathbf{F}^*(\mathbf{f}, \mathbf{g}) = \mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{F}_\uparrow(\mathbf{g}))$$

which is non-degenerate. In addition, if $\mathbf{F}$ is symmetric/anti-symmetric/positive definite/negative definite/indefinite then so is $\mathbf{F}^*$.

Proof. Since $\mathbf{F}$ is bilinear (i.e., linear in both positions) then $\mathbf{F}^*$ is bilinear.

Suppose $\mathbf{F}$ is non-degenerate. Let $\mathbf{f} \in X^*$ be such that $\mathbf{F}^*(\mathbf{F}, \mathbf{g}) = 0$ for all $\mathbf{g} \in X^*$. Then $\mathbf{F}^*(\mathbf{F}, \mathbf{g}) = \mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{F}_\uparrow(\mathbf{g})) = 0$ for all $\mathbf{g} \in X^*$. Since $\mathbf{F}_\uparrow : X^* \to X$ is onto, then we can equivalently say $\mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{y}) = 0$ for all $\mathbf{y} \in X$ (we take $\mathbf{y} = \mathbf{F}_\uparrow(\mathbf{g})$). Since $\mathbf{F}$ is non-degenerate, this implies $\mathbf{F}_\uparrow(\mathbf{f}) = 0$ and since $\mathbf{F}_\uparrow$ is linear and one to one, this implies that $\mathbf{f} = \mathbf{0}$. Therefore $\mathbf{F}_\uparrow$ is non-degenerate, as claimed.

Lemma IV.1.11

Lemma IV.1.11. A non-degenerate bilinear form $\mathbf{F}$ on a vector space X induces a bilinear form $\mathbf{F}^*$ on X^* by

$$\mathbf{F}^*(\mathbf{f},\mathbf{g}) = \mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}),\mathbf{F}_\uparrow(\mathbf{g}))$$

which is non-degenerate. In addition, if $\mathbf{F}$ is symmetric/anti-symmetric/positive definite/negative definite/indefinite then so is $\mathbf{F}^*$.

Proof. Since $\mathbf{F}$ is bilinear (i.e., linear in both positions) then $\mathbf{F}^*$ is bilinear.

Suppose $\mathbf{F}$ is non-degenerate. Let $\mathbf{f} \in X^*$ be such that $\mathbf{F}^*(\mathbf{F},\mathbf{g}) = 0$ for all $\mathbf{g} \in X^*$. Then $\mathbf{F}^*(\mathbf{F},\mathbf{g}) = \mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}),\mathbf{F}_\uparrow(\mathbf{g})) = 0$ for all $\mathbf{g} \in X^*$. Since $\mathbf{F}_\uparrow : X^* \to X$ is onto, then we can equivalently say $\mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}),\mathbf{y}) = 0$ for all $\mathbf{y} \in X$ (we take $\mathbf{y} = \mathbf{F}_\uparrow(\mathbf{g})$). Since $\mathbf{F}$ is non-degenerate, this implies $\mathbf{F}_\uparrow(\mathbf{f}) = 0$ and since $\mathbf{F}_\uparrow$ is linear and one to one, this implies that $\mathbf{f} = \mathbf{0}$. Therefore $\mathbf{F}_\uparrow$ is non-degenerate, as claimed.

The claims about symmetry/etc. are to be shown in Exercise IV.1.8. □

Lemma IV.1.11

Lemma IV.1.11. A non-degenerate bilinear form $\mathbf{F}$ on a vector space X induces a bilinear form $\mathbf{F}^*$ on X^* by

$$\mathbf{F}^*(\mathbf{f}, \mathbf{g}) = \mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{F}_\uparrow(\mathbf{g}))$$

which is non-degenerate. In addition, if $\mathbf{F}$ is symmetric/anti-symmetric/positive definite/negative definite/indefinite then so is $\mathbf{F}^*$.

Proof. Since $\mathbf{F}$ is bilinear (i.e., linear in both positions) then $\mathbf{F}^*$ is bilinear.

Suppose $\mathbf{F}$ is non-degenerate. Let $\mathbf{f} \in X^*$ be such that $\mathbf{F}^*(\mathbf{F}, \mathbf{g}) = 0$ for all $\mathbf{g} \in X^*$. Then $\mathbf{F}^*(\mathbf{F}, \mathbf{g}) = \mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{F}_\uparrow(\mathbf{g})) = 0$ for all $\mathbf{g} \in X^*$. Since $\mathbf{F}_\uparrow : X^* \to X$ is onto, then we can equivalently say $\mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{y}) = 0$ for all $\mathbf{y} \in X$ (we take $\mathbf{y} = \mathbf{F}_\uparrow(\mathbf{g})$). Since $\mathbf{F}$ is non-degenerate, this implies $\mathbf{F}_\uparrow(\mathbf{f}) = 0$ and since $\mathbf{F}_\uparrow$ is linear and one to one, this implies that $\mathbf{f} = \mathbf{0}$. Therefore $\mathbf{F}_\uparrow$ is non-degenerate, as claimed.

The claims about symmetry/etc. are to be shown in Exercise IV.1.8. □