Differential Geometry

Chapter IV. Metric Vector Spaces

IV.2. Maps—Proofs of Theorems

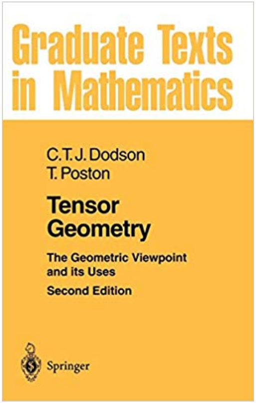


Table of contents

Theorem IV.2.01

Theorem IV.2.01. Let S be a non-degenerate subspace of a metric vector space X. Then there is a unique linear operator $\mathbf{P} : X \to S$ such that $(\mathbf{x} - \mathbf{Rx}) \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in S$.

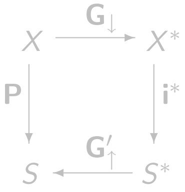
Proof. Let $\mathbf{G}$ be the metric tensor on X. Let $\mathbf{G}'$ be the metric tensor on subspace S induced by $\mathbf{G}$ (so $\mathbf{G}' = \mathbf{G}|_S$). Let $\mathbf{i} : S \to X$ be the inclusion map embedding (the "inclusion map") S into X. Then the dual map $\mathbf{i}^* : X^* \to S^*$ defined as $\mathbf{i}^*(\mathbf{f}) = \mathbf{f}|_S = \mathbf{f} \circ \mathbf{i} \in S^*$ for $\mathbf{F} \in X^*$.

Theorem IV.2.01

Theorem IV.2.01. Let S be a non-degenerate subspace of a metric vector space X. Then there is a unique linear operator $\mathbf{P} : X \to S$ such that $(\mathbf{x} - \mathbf{Rx}) \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in S$.

Proof. Let $\mathbf{G}$ be the metric tensor on X. Let $\mathbf{G}'$ be the metric tensor on subspace S induced by $\mathbf{G}$ (so $\mathbf{G}' = \mathbf{G}|_S$). Let $\mathbf{i} : S \to X$ be the inclusion map embedding (the "inclusion map") S into X. Then the dual map $\mathbf{i}^* : X^* \to S^*$ defined as $\mathbf{i}^*(\mathbf{f}) = \mathbf{f}|_S = \mathbf{f} \circ \mathbf{i} \in S^*$ for $\mathbf{F} \in X^*$. We set $\mathbf{P} = (\mathbf{G}'_\downarrow)^{-1}$ ($\mathbf{P}$ is linear since it is a composition of linear mappings) with $\mathbf{G}'_\downarrow(\mathbf{s} = \mathbf{s}^*$ where $\mathbf{s}^*(\mathbf{y}) = \mathbf{G}'(\mathbf{s}, \mathbf{y})$ for all $\mathbf{y} \in S$ (see Theorem IV.1.09) and $\mathbf{G}_\downarrow : S \to X^*$ is defined as $\mathbf{G}_\downarrow(\mathbf{x}) = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y}) = \mathbf{G}(\mathbf{x}, \mathbf{y})$ (see Theorem IV.1.09).

We then have the diagram:

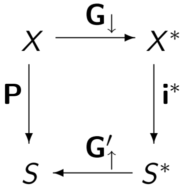


Theorem IV.2.01

Theorem IV.2.01. Let S be a non-degenerate subspace of a metric vector space X. Then there is a unique linear operator $\mathbf{P} : X \to S$ such that $(\mathbf{x} - \mathbf{Rx}) \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in S$.

Proof. Let $\mathbf{G}$ be the metric tensor on X. Let $\mathbf{G}'$ be the metric tensor on subspace S induced by $\mathbf{G}$ (so $\mathbf{G}' = \mathbf{G}|_S$). Let $\mathbf{i} : S \to X$ be the inclusion map embedding (the "inclusion map") S into X. Then the dual map $\mathbf{i}^* : X^* \to S^*$ defined as $\mathbf{i}^*(\mathbf{f}) = \mathbf{f}|_S = \mathbf{f} \circ \mathbf{i} \in S^*$ for $\mathbf{F} \in X^*$. We set $\mathbf{P} = (\mathbf{G}'_{\downarrow})^{-1}$ ($\mathbf{P}$ is linear since it is a composition of linear mappings) with $\mathbf{G}'_{\downarrow}(\mathbf{s} = \mathbf{s}^*$ where $\mathbf{s}^*(\mathbf{y}) = \mathbf{G}'(\mathbf{s}, \mathbf{y})$ for all $\mathbf{y} \in S$ (see Theorem IV.1.09) and $\mathbf{G}_{\downarrow} : S \to X^*$ is defined as $\mathbf{G}_{\downarrow}(\mathbf{x}) = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y}) = \mathbf{G}(\mathbf{x}, \mathbf{y})$ (see Theorem IV.1.09).

We then have the diagram:



Theorem IV.2.01 (continued 1)

Proof continued. For any $\mathbf{y} \in S$ we have

$$\begin{aligned}(\mathbf{x} - \mathbf{Px}) \cdot \mathbf{y} &= \mathbf{G}(\mathbf{x} - \mathbf{Px}, \mathbf{y}) = \mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{G}(\mathbf{Px}, \mathbf{y}) \\ &= \mathbf{G}(\mathbf{x}, \mathbf{y}) - (\mathbf{Px}) \cdot \mathbf{y} \text{ where the dot product is in } X \\ &\quad \text{and so is based on } \mathbf{G} \\ &= \mathbf{G}(\mathbf{x}, \mathbf{y}) - \mathbf{G}'_{\uparrow}(\mathbf{i}^*(\mathbf{G}_{\downarrow}\mathbf{x})) \cdot \mathbf{y} \text{ by the definition of } \mathbf{P} \\ &= \mathbf{G}(\mathbf{x}, \mathbf{y}) - \mathbf{G}'(\mathbf{G}'_{\uparrow}(\mathbf{i}^*(\mathbf{G}_{\downarrow}\mathbf{x})), \mathbf{y}) \text{ since the dot product is} \\ &\quad \text{in } S \text{ and so is based on } \mathbf{G}. \qquad (*)\end{aligned}$$

Now $\mathbf{G}_{\downarrow}\mathbf{x} \in X^*$, say $\mathbf{G}_{\downarrow}\mathbf{x} = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y}) = \mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{y} \in X$. Next, $\mathbf{i}(\mathbf{G}_{\downarrow}\mathbf{x}) \in S^*$, say $\mathbf{i}^*(\mathbf{x}^*) = \mathbf{s}^*$ where $\mathbf{s}^*(\mathbf{y}) = \mathbf{x}^*(\mathbf{y}) = \mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{y} \in S \subset X$ (so $\mathbf{s}^* = \mathbf{x}^*|_S$). Also, $\mathbf{G}_{\uparrow}(\mathbf{i}^*(\mathbf{G}_{\downarrow}\mathbf{x}) = \mathbf{G}'_{\uparrow}(\mathbf{s}^*) = \mathbf{s} \in S$ where $\mathbf{s}^*(\mathbf{y}) = \mathbf{G}'(\mathbf{s}, \mathbf{y}) = \mathbf{s} \cdot \mathbf{y}$ for all $\mathbf{y} \in S$. But $\mathbf{s}^* = \mathbf{x}^*|S$, so $\mathbf{G}'(\mathbf{s}, \mathbf{y}) = \mathbf{s} \cdot \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{y} \in S$.

Theorem IV.2.01 (continued 1)

Proof continued. For any $\mathbf{y} \in S$ we have

$$\begin{aligned}(\mathbf{x}-\mathbf{Px})\cdot\mathbf{y} &= \mathbf{G}(\mathbf{x}-\mathbf{Px},\mathbf{y}) = \mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{G}(\mathbf{Px},\mathbf{y}) \\ &= \mathbf{G}(\mathbf{x},\mathbf{y}) - (\mathbf{Px})\cdot\mathbf{y} \text{ where the dot product is in } X \\ &\quad \text{and so is based on } \mathbf{G} \\ &= \mathbf{G}(\mathbf{x},\mathbf{y}) - \mathbf{G}'_{\uparrow}(\mathbf{i}^*(\mathbf{G}_{\downarrow}\mathbf{x}))\cdot\mathbf{y} \text{ by the definition of } \mathbf{P} \\ &= \mathbf{G}(\mathbf{x},\mathbf{y}) - \mathbf{G}'(\mathbf{G}'_{\uparrow}(\mathbf{i}^*(\mathbf{G}_{\downarrow}\mathbf{x})),\mathbf{y}) \text{ since the dot product is} \\ &\quad \text{in } S \text{ and so is based on } \mathbf{G}. \qquad (*)\end{aligned}$$

Now $\mathbf{G}_{\downarrow}\mathbf{x} \in X^*$, say $\mathbf{G}_{\downarrow}\mathbf{x} = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y}) = \mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$ for all $\mathbf{y} \in X$. Next, $\mathbf{i}(\mathbf{G}_{\downarrow}\mathbf{x}) \in S^*$, say $\mathbf{i}^*(\mathbf{x}^*) = \mathbf{s}^*$ where $\mathbf{s}^*(\mathbf{y}) = \mathbf{x}^*(\mathbf{y}) = \mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$ for all $\mathbf{y} \in S \subset X$ (so $\mathbf{s}^* = \mathbf{x}^*|_S$). Also, $\mathbf{G}_{\uparrow}(\mathbf{i}^*(\mathbf{G}_{\downarrow}\mathbf{x}) = \mathbf{G}'_{\uparrow}(\mathbf{s}^*) = \mathbf{s} \in S$ where $\mathbf{s}^*(\mathbf{y}) = \mathbf{G}'(\mathbf{s},\mathbf{y}) = \mathbf{s}\cdot\mathbf{y}$ for all $\mathbf{y} \in S$. But $\mathbf{s}^* = \mathbf{x}^*|S$, so $\mathbf{G}'(\mathbf{s},\mathbf{y}) = \mathbf{s}\cdot\mathbf{y} = \mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$ for all $\mathbf{y} \in S$.

Theorem IV.2.01 (continued 2)

Proof continued. So

$$\begin{aligned}\mathbf{G}'(\mathbf{G}'_{\uparrow}(\mathbf{i}^*(\mathbf{G}_{\downarrow}\mathbf{x}), \mathbf{y}) &= \mathbf{G}'(\mathbf{s}, \mathbf{y}) \\ &= \mathbf{s} \cdot \mathbf{y} \text{ where the dot product is in } S \\ &= \mathbf{x} \cdot \mathbf{y} \text{ where the dot product is in } X \\ &= \mathbf{G}(\mathbf{x} \cdot \mathbf{y}) \text{ since the dot product in } X \\ &\quad \text{ is based on metric tensor } G.\end{aligned}$$

The by $(*)$ we now have $(\mathbf{x} - \mathbf{Px}) \cdot \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{G}(\mathbf{x}, \mathbf{y}) = 0$. So there exists a linear map $\mathbf{P}$ as claimed.

For uniqueness, suppose $\mathbf{Q} : X \to S$ is a linear map such that $(\mathbf{x} - \mathbf{Qx}) \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in S$ and $\mathbf{x} \in X$. Then, for all $\mathbf{y} \in S$ and $\mathbf{x} \in X$ we have

$$\begin{aligned}(\mathbf{x} - \mathbf{Px} - (\mathbf{x} - \mathbf{Qx})) \cdot \mathbf{y} &= \mathbf{G}(\mathbf{x} - \mathbf{Px} - (\mathbf{x} - \mathbf{Qx}), \mathbf{y}) \\ &= \mathbf{G}(\mathbf{Qx} - \mathbf{Px}, \mathbf{y}) = (\mathbf{Qx} - \mathbf{Px}) \cdot \mathbf{y} = 0.\end{aligned}$$

Theorem IV.2.01 (continued 2)

Proof continued. So

$$\begin{aligned} \mathbf{G}'(\mathbf{G}'_{\uparrow}(\mathbf{i}^*(\mathbf{G}_{\downarrow}\mathbf{x}), \mathbf{y}) &= \mathbf{G}'(\mathbf{s}, \mathbf{y}) \\ &= \mathbf{s} \cdot \mathbf{y} \text{ where the dot product is in } S \\ &= \mathbf{x} \cdot \mathbf{y} \text{ where the dot product is in } X \\ &= \mathbf{G}(\mathbf{x} \cdot \mathbf{y}) \text{ since the dot product in } X \\ &\quad \text{ is based on metric tensor } G. \end{aligned}$$

The by $(*)$ we now have $(\mathbf{x} - \mathbf{P}\mathbf{x}) \cdot \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{G}(\mathbf{x}, \mathbf{y}) = 0$. So there exists a linear map $\mathbf{P}$ as claimed.

For uniqueness, suppose $\mathbf{Q} : X \to S$ is a linear map such that $(\mathbf{x} - \mathbf{Q}\mathbf{x}) \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in S$ and $\mathbf{x} \in X$. Then, for all $\mathbf{y} \in S$ and $\mathbf{x} \in X$ we have

$$\begin{aligned} (\mathbf{x} - \mathbf{P}\mathbf{x} - (\mathbf{x} - \mathbf{Q}\mathbf{x})) \cdot \mathbf{y} &= \mathbf{G}(\mathbf{x} - \mathbf{P}\mathbf{x} - (\mathbf{x} - \mathbf{Q}\mathbf{x}), \mathbf{y}) \\ &= \mathbf{G}(\mathbf{Q}\mathbf{x} - \mathbf{P}\mathbf{x}, \mathbf{y}) = (\mathbf{Q}\mathbf{x} - \mathbf{P}\mathbf{x}) \cdot \mathbf{y} = 0. \end{aligned}$$

Theorem IV.2.01 (continued 3)

Theorem IV.2.01. Let S be a non-degenerate subspace of a metric vector space X. Then there is a unique linear operator $\mathbf{P} : X \to S$ such that $(\mathbf{x} - \mathbf{Rx}) \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in S$.

Proof continued. Since $\mathbf{G}$ is non-degenerate on S by hypothesis (that is, S is a non-degenerate subspace of X), and $\mathbf{Qx} - \mathbf{Px} \in S$ (since $\mathbf{Qx}$, $\mathbf{Px} \in S$ by hypothesis, since $\mathbf{P}, \mathbf{Q} : X \to S$), then we must have $\mathbf{Qx} = \mathbf{Px}$ for all $\mathbf{x} \in X$. That is, $\mathbf{P} = \mathbf{Q}$ and the linear map is unique. □

Corollary IV.2.02

Corollary IV.2.02. The projection operator $\mathbf{P}$ onto S is idempotent. That is, $\mathbf{P}(\mathbf{P}\mathbf{x}) = \mathbf{P}\mathbf{x}$ for all $\mathbf{x} \in X$.

Proof. Let $\mathbf{x} \in X$ and say $\mathbf{y} = \mathbf{P}\mathbf{x}$. Since $\mathbf{P}$ is a projection operator, then $(\mathbf{y} = \mathbf{P}\mathbf{x}) \cdot \mathbf{y}' = 0$ for all $\mathbf{y}' \in S$. But $\mathbf{y} = \mathbf{P}\mathbf{x} \in S$ since $\mathbf{P} : X \to S$ and $\mathbf{G}$ is non-degenerate on S (by definition) so $\mathbf{y} = \mathbf{P}\mathbf{y} = \mathbf{0}$ and $\mathbf{y} = \mathbf{P}\mathbf{y}$. Therefore $\mathbf{y} = \mathbf{P}\mathbf{x} = \mathbf{P}\mathbf{y} = \mathbf{P}(\mathbf{P}\mathbf{y})$. Since $\mathbf{x} \in X$ is arbitrary, then $\mathbf{P}\mathbf{x} = \mathbf{P}(\mathbf{P}\mathbf{x})$ for all $\mathbf{x} \in X$, as claimed. □

Corollary IV.2.02

Corollary IV.2.02. The projection operator $\mathbf{P}$ onto S is idempotent. That is, $\mathbf{P}(\mathbf{P}\mathbf{x}) = \mathbf{P}\mathbf{x}$ for all $\mathbf{x} \in X$.

Proof. Let $\mathbf{x} \in X$ and say $\mathbf{y} = \mathbf{P}\mathbf{x}$. Since $\mathbf{P}$ is a projection operator, then $(\mathbf{y} = \mathbf{P}\mathbf{x}) \cdot \mathbf{y}' = 0$ for all $\mathbf{y}' \in S$. But $\mathbf{y} = \mathbf{P}\mathbf{x} \in S$ since $\mathbf{P} : X \to S$ and $\mathbf{G}$ is non-degenerate on S (by definition) so $\mathbf{y} = \mathbf{P}\mathbf{y} = \mathbf{0}$ and $\mathbf{y} = \mathbf{P}\mathbf{y}$. Therefore $\mathbf{y} = \mathbf{P}\mathbf{x} = \mathbf{P}\mathbf{y} = \mathbf{P}(\mathbf{P}\mathbf{y})$. Since $\mathbf{x} \in X$ is arbitrary, then $\mathbf{P}\mathbf{x} = \mathbf{P}(\mathbf{P}\mathbf{x})$ for all $\mathbf{x} \in X$, as claimed. □

Lemma IV.2.04

Lemma IV.2.04. For any non-degenerate subspace S of X, each $x \in X$ can be uniquely expressed as $\mathbf{x} = \mathbf{s} + \mathbf{t}$ where $\mathbf{x} \in S$ and $\mathbf{f} \in S^\perp$.

Proof. Let $\mathbf{P}$ be the orthogonal projection onto S. Set $\mathbf{s} = \mathbf{Px}$ and $\mathbf{t} = \mathbf{x} - \mathbf{Px}$. Then $\mathbf{s} \in S$ and $\mathbf{t} \in S^\perp$ by the definition of $\mathbf{P}$, and $\mathbf{x} = \mathbf{s} + \mathbf{t}$. Since $\mathbf{P}$ is the unique orthogonal projection onto S by Theorem IV.2.01, then $\mathbf{s} = \mathbf{Px}$ is unique and so $\mathbf{t} = \mathbf{x} - \mathbf{Px}$ is unique, as claimed. □

Lemma IV.2.04

Lemma IV.2.04. For any non-degenerate subspace S of X, each $x \in X$ can be uniquely expressed as $\mathbf{x} = \mathbf{s} + \mathbf{t}$ where $\mathbf{x} \in S$ and $\mathbf{f} \in S^{\perp}$.

Proof. Let $\mathbf{P}$ be the orthogonal projection onto S. Set $\mathbf{s} = \mathbf{Px}$ and $\mathbf{t} = \mathbf{x} - \mathbf{Px}$. Then $\mathbf{s} \in S$ and $\mathbf{t} \in S^{\perp}$ by the definition of $\mathbf{P}$, and $\mathbf{x} = \mathbf{s} + \mathbf{t}$. Since $\mathbf{P}$ is the unique orthogonal projection onto S by Theorem IV.2.01, then $\mathbf{s} = \mathbf{Px}$ is unique and so $\mathbf{t} = \mathbf{x} - \mathbf{Px}$ is unique, as claimed. □

Corollary IV.2.06

Corollary IV.2.06. If G is non-degenerate on S, it is non-degenerate on $S^\perp$.

Proof. Recall that by definition, **G** in non-degenerate on S if $\mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = 0$ for all $y \in S$ implies $\mathbf{x} = \mathbf{0}$. Let $\mathbf{x} \in S^\perp$. Then $\mathbf{x} \cdot \mathbf{t} + \mathbf{x} \cdot \mathbf{s} = 0$ for all $\mathbf{t} \in S^\perp$ and $\mathbf{s} \in S$, or $\mathbf{x} \cdot (\mathbf{t} + \mathbf{s}) = 0$ for all $\mathbf{t} \in S^\perp$ and $\mathbf{s} \in S$.

Corollary IV.2.06

Corollary IV.2.06. If G is non-degenerate on S, it is non-degenerate on $S^\perp$.

Proof. Recall that by definition, $\mathbf{G}$ in non-degenerate on S if $\mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{x}\cdot\mathbf{y} = 0$ for all $y \in S$ implies $\mathbf{x} = \mathbf{0}$. Let $\mathbf{x} \in S^\perp$. Then $\mathbf{x}\cdot\mathbf{t} + \mathbf{x}\cdot\mathbf{s} = 0$ for all $\mathbf{t} \in S^\perp$ and $\mathbf{s} \in S$, or $\mathbf{x}\cdot(\mathbf{t}+\mathbf{s}) = 0$ for all $\mathbf{t} \in S^\perp$ and $\mathbf{s} \in S$. Since by Lemma IV.2.04, every element $\mathbf{y}$ of X is of the form $\mathbf{y} = \mathbf{s} + \mathbf{t}$ then we have $\mathbf{x}\cdot\mathbf{y} = \mathbf{G}(\mathbf{x},\mathbf{y}) = 0$ for all $\mathbf{y} \in X$. Since $\mathbf{G}$ is non-degenerate on X then this implies $\mathbf{x} = \mathbf{0}$. Therefore $\mathbf{G}$ is non-degenerate on $S^\perp$, as claimed. □

Corollary IV.2.06

Corollary IV.2.06. If G is non-degenerate on S, it is non-degenerate on $S^\perp$.

Proof. Recall that by definition, $\mathbf{G}$ in non-degenerate on S if $\mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = 0$ for all $y \in S$ implies $\mathbf{x} = \mathbf{0}$. Let $\mathbf{x} \in S^\perp$. Then $\mathbf{x} \cdot \mathbf{t} + \mathbf{x} \cdot \mathbf{s} = 0$ for all $\mathbf{t} \in S^\perp$ and $\mathbf{s} \in S$, or $\mathbf{x} \cdot (\mathbf{t} + \mathbf{s}) = 0$ for all $\mathbf{t} \in S^\perp$ and $\mathbf{s} \in S$. Since by Lemma IV.2.04, every element $\mathbf{y}$ of X is of the form $\mathbf{y} = \mathbf{s} + \mathbf{t}$ then we have $\mathbf{x} \cdot \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{y} \in X$. Since $\mathbf{G}$ is non-degenerate on X then this implies $\mathbf{x} = \mathbf{0}$. Therefore $\mathbf{G}$ is non-degenerate on $S^\perp$, as claimed. □

Lemma IV.2.A

Lemma IV.2.A. Properties of Adjoint.
For $\mathbf{A}$ and $\mathbf{B}$ linear operators on a metric vector space (X, G) we have:

(a) $\mathbf{I}^T = \mathbf{I}$ where $\mathbf{I}$ is the identity operator.

(b) $(\mathbf{A}^T)^T = \mathbf{A}$.

(c) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Proof. (a) We have $\mathbf{Ix} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{Iy}$ for all $\mathbf{x}, \mathbf{y} \in X$, so $\mathbf{I}^T = \mathbf{I}$ (since the transpose of a linear operator is unique).

Lemma IV.2.A

Lemma IV.2.A. Properties of Adjoint.
For $\mathbf{A}$ and $\mathbf{B}$ linear operators on a metric vector space (X, G) we have:

(a) $\mathbf{I}^T = \mathbf{I}$ where $\mathbf{I}$ is the identity operator.
(b) $(\mathbf{A}^T)^T = \mathbf{A}$.
(c) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Proof. (a) We have $\mathbf{Ix} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{Iy}$ for all $\mathbf{x}, \mathbf{y} \in X$, so $\mathbf{I}^T = \mathbf{I}$ (since the transpose of a linear operator is unique).

(b) By the symmetry of $\mathbf{G}$ (and so the symmetry of the dot product) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{y} \cdot \mathbf{A}^T\mathbf{x} = \mathbf{A}^T\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{Ay}$ and $\mathbf{y} \cdot \mathbf{A}^T\mathbf{x} = (\mathbf{A}^T)^T\mathbf{y} \cdot \mathbf{x} = \mathbf{x} \cdot (\mathbf{A}^T)^T\mathbf{y}$. So $\mathbf{x} \cdot \mathbf{Ay} = \mathbf{x}(\mathbf{A}^T)^T\mathbf{y}$ or $\mathbf{x} \cdot (\mathbf{Ay} - (\mathbf{A}^T)^T\mathbf{y}) = 0$ for all $\mathbf{x} \in X$. So the non-degeneracy of $\mathbf{G}$ implies $\mathbf{Ay} - (\mathbf{A}^T)^T\mathbf{y} = \mathbf{0}$, or $\mathbf{Ay} = (\mathbf{A}^T)^T\mathbf{y}$. Since this holds for all $\mathbf{y} \in X$ then $\mathbf{A} = (\mathbf{A}^T)^T$.

Lemma IV.2.A

Lemma IV.2.A. Properties of Adjoint.
For $\mathbf{A}$ and $\mathbf{B}$ linear operators on a metric vector space (X, G) we have:

(a) $\mathbf{I}^T = \mathbf{I}$ where $\mathbf{I}$ is the identity operator.

(b) $(\mathbf{A}^T)^T = \mathbf{A}$.

(c) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Proof. (a) We have $\mathbf{Ix} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{Iy}$ for all $\mathbf{x}, \mathbf{y} \in X$, so $\mathbf{I}^T = \mathbf{I}$ (since the transpose of a linear operator is unique).

(b) By the symmetry of $\mathbf{G}$ (and so the symmetry of the dot product) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{y} \cdot \mathbf{A}^T\mathbf{x} = \mathbf{A}^T\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{Ay}$ and $\mathbf{y} \cdot \mathbf{A}^T\mathbf{x} = (\mathbf{A}^T)^T\mathbf{y} \cdot \mathbf{x} = \mathbf{x} \cdot (\mathbf{A}^T)^T\mathbf{y}$. So $\mathbf{x} \cdot \mathbf{Ay} = \mathbf{x}(\mathbf{A}^T)^T\mathbf{y}$ or $\mathbf{x} \cdot (\mathbf{Ay} - (\mathbf{A}^T)^T\mathbf{y}) = 0$ for all $\mathbf{x} \in X$. So the non-degeneracy of $\mathbf{G}$ implies $\mathbf{Ay} - (\mathbf{A}^T)^T\mathbf{y} = \mathbf{0}$, or $\mathbf{Ay} = (\mathbf{A}^T)^T\mathbf{y}$. Since this holds for all $\mathbf{y} \in X$ then $\mathbf{A} = (\mathbf{A}^T)^T$.

Lemma IV.2.A (continued)

Lemma IV.2.A. Properties of Adjoint.
For $\mathbf{A}$ and $\mathbf{B}$ linear operators on a metric vector space (X, G) we have:

(a) $\mathbf{I}^T = \mathbf{I}$ where $\mathbf{I}$ is the identity operator.
(b) $(\mathbf{A}^T)^T = \mathbf{A}$.
(c) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Proof (continued). (c) Notice that $\mathbf{AB}$ is also a linear operator on $(X, \mathbf{G})$. For all $\mathbf{x}, \mathbf{y} \in X$ we have

$$(\mathbf{AB})^T\mathbf{x}\cdot\mathbf{y} = \mathbf{x}\cdot(\mathbf{ABy}) = \mathbf{A}^T\mathbf{x}\cdot\mathbf{By} = \mathbf{B}^T\mathbf{A}^T\mathbf{x}\cdot\mathbf{y}.$$

So $(\mathbf{AB})^T\mathbf{x}\cdot\mathbf{y} - \mathbf{B}^T\mathbf{A}^T\mathbf{x}\cdot\mathbf{y} = ((\mathbf{AB})^T - \mathbf{B}^T\mathbf{A}^T\mathbf{x})\cdot\mathbf{y} = 0$ for all $\mathbf{y} \in X$. So the non-degenerate of $\mathbf{G}$ implies $(\mathbf{AB})^T\mathbf{x} - \mathbf{B}^T\mathbf{A}^T\mathbf{x} = \mathbf{0}$ or $(\mathbf{AB})^T\mathbf{x} = \mathbf{B}^T\mathbf{A}^T\mathbf{x} = \mathbf{B}^T\mathbf{A}^T = \mathbf{x}$. Since this holds for all $\mathbf{x} \in X$ then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. □

Lemma IV.2.A (continued)

Lemma IV.2.A. Properties of Adjoint.
For $\mathbf{A}$ and $\mathbf{B}$ linear operators on a metric vector space (X, G) we have:

(a) $\mathbf{I}^T = \mathbf{I}$ where $\mathbf{I}$ is the identity operator.

(b) $(\mathbf{A}^T)^T = \mathbf{A}$.

(c) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Proof (continued). (c) Notice that $\mathbf{AB}$ is also a linear operator on $(X, \mathbf{G})$. For all $\mathbf{x}, \mathbf{y} \in X$ we have

$$(\mathbf{AB})^T\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot (\mathbf{ABy}) = \mathbf{A}^T\mathbf{x} \cdot \mathbf{By} = \mathbf{B}^T\mathbf{A}^T\mathbf{x} \cdot \mathbf{y}.$$

So $(\mathbf{AB})^T\mathbf{x} \cdot \mathbf{y} - \mathbf{B}^T\mathbf{A}^T\mathbf{x} \cdot \mathbf{y} = ((\mathbf{AB})^T - \mathbf{B}^T\mathbf{A}^T\mathbf{x}) \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in X$. So the non-degenerate of $\mathbf{G}$ implies $(\mathbf{AB})^T\mathbf{x} - \mathbf{B}^T\mathbf{A}^T\mathbf{x} = \mathbf{0}$ or $(\mathbf{AB})^T\mathbf{x} = \mathbf{B}^T\mathbf{A}^T\mathbf{x} = \mathbf{B}^T\mathbf{A}^T = \mathbf{x}$. Since this holds for all $\mathbf{x} \in X$ then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. □

Lemma IV.2.09

Lemma IV.2.09. An operator $\mathbf{A}$ on a metric vector space $(X, \mathbf{G})$ is orthogonal if and only if $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.

Proof. Let $\mathbf{x} \in X$. Then $\mathbf{A}\mathbf{x} \cdot \mathbf{A}\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{y} \in X$ (that is, $\mathbf{A}$ is orthogonal) if and only if $(\mathbf{A}^T\mathbf{A}\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ by the definition of adjoint. This is equivalent to $(\mathbf{A}^T\mathbf{A}\mathbf{x}) \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y} = \mathbf{0}$ or $(\mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{x}) \cdot \mathbf{y} = \mathbf{0}$ for all $\mathbf{y} \in X$. Since $\mathbf{G}$ is non-degenerate (by the definition of metric tensor) then this is equivalent to $\mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{x} = \mathbf{0}$ or, since $\mathbf{x}$ is an arbitrary element of X, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. □

Lemma IV.2.09

Lemma IV.2.09. An operator $\mathbf{A}$ on a metric vector space $(X, \mathbf{G})$ is orthogonal if and only if $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.

Proof. Let $\mathbf{x} \in X$. Then $\mathbf{A}\mathbf{x} \cdot \mathbf{A}\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{y} \in X$ (that is, $\mathbf{A}$ is orthogonal) if and only if $(\mathbf{A}^T\mathbf{A}\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ by the definition of adjoint. This is equivalent to $(\mathbf{A}^T\mathbf{A}\mathbf{x}) \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y} = \mathbf{0}$ or $(\mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{x}) \cdot \mathbf{y} = \mathbf{0}$ for all $\mathbf{y} \in X$. Since $\mathbf{G}$ is non-degenerate (by the definition of metric tensor) then this is equivalent to $\mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{x} = \mathbf{0}$ or, since $\mathbf{x}$ is an arbitrary element of X, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. □

Lemma IV.2.11

Lemma IV.2.11. Orthogonal projection $\mathbf{P}$ onto a non-degenerate subspace S of a metric vector space X is a self adjoint operator.

Proof. Let $\mathbf{x} = \mathbf{s} + \mathbf{t}$ and $\mathbf{y} = \mathbf{s}' + \mathbf{t}'$ where $\mathbf{s}, \mathbf{s}' \in S$ and $\mathbf{t}, \mathbf{t}' \in S^\perp$ (by Lemma IV.2.04, the choices of $\mathbf{s}, \mathbf{s}', \mathbf{t}, \mathbf{t}'$ are unique). Then

$$\begin{aligned}
\mathbf{Px} \cdot \mathbf{y} &= \mathbf{P}(\mathbf{s} + \mathbf{t}) \cdot (\mathbf{s}' + \mathbf{t}') \\
&= \mathbf{s} \cdot (\mathbf{s}' + \mathbf{t}') \text{ since } \mathbf{Px} = \mathbf{P}(\mathbf{s} + \mathbf{t}) = \mathbf{s} \\
&= \mathbf{s} \cdot \mathbf{s}' + \mathbf{s} \cdot \mathbf{t}' \\
&= \mathbf{s} \cdot \mathbf{s}' \text{ since } \mathbf{s} \cdot \mathbf{t}' = 0 \text{ because } \mathbf{t}' \in S^\perp.
\end{aligned}$$

Lemma IV.2.11

Lemma IV.2.11. Orthogonal projection $\mathbf{P}$ onto a non-degenerate subspace S of a metric vector space X is a self adjoint operator.

Proof. Let $\mathbf{x} = \mathbf{s} + \mathbf{t}$ and $\mathbf{y} = \mathbf{s}' + \mathbf{t}'$ where $\mathbf{s}, \mathbf{s}' \in S$ and $\mathbf{t}, \mathbf{t}' \in S^\perp$ (by Lemma IV.2.04, the choices of $\mathbf{s}, \mathbf{s}', \mathbf{t}, \mathbf{t}'$ are unique). Then

$$\begin{aligned} \mathbf{Px} \cdot \mathbf{y} &= \mathbf{P}(\mathbf{s} + \mathbf{t}) \cdot (\mathbf{s}' + \mathbf{t}') \\ &= \mathbf{s} \cdot (\mathbf{s}' + \mathbf{t}') \text{ since } \mathbf{Px} = \mathbf{P}(\mathbf{s} + \mathbf{t}) = \mathbf{s} \\ &= \mathbf{s} \cdot \mathbf{s}' + \mathbf{s} \cdot \mathbf{t}' \\ &= \mathbf{s} \cdot \mathbf{s}' \text{ since } \mathbf{s} \cdot \mathbf{t}' = 0 \text{ because } \mathbf{t}' \in S^\perp. \end{aligned}$$

Similarly,

$$\mathbf{x} \cdot \mathbf{Py} = (\mathbf{s} + \mathbf{t}) \cdot \mathbf{P}(\mathbf{s}' + \mathbf{t}') = (\mathbf{s} + \mathbf{t}) \cdot \mathbf{s}' = \mathbf{s} \cdot \mathbf{s}' + \mathbf{t} \cdot \mathbf{s}' = \mathbf{s} \cdot \mathbf{s}',$$

so that $\mathbf{Px} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{Py}$. Since this holds for all $\mathbf{x}, \mathbf{y} \in X$, then $\mathbf{P}^T = \mathbf{P}$ and $\mathbf{P}$ is self adjoint, as claimed. □

Lemma IV.2.11

Lemma IV.2.11. Orthogonal projection $\mathbf{P}$ onto a non-degenerate subspace S of a metric vector space X is a self adjoint operator.

Proof. Let $\mathbf{x} = \mathbf{s} + \mathbf{t}$ and $\mathbf{y} = \mathbf{s}' + \mathbf{t}'$ where $\mathbf{s}, \mathbf{s}' \in S$ and $\mathbf{t}, \mathbf{t}' \in S^\perp$ (by Lemma IV.2.04, the choices of $\mathbf{s}, \mathbf{s}', \mathbf{t}, \mathbf{t}'$ are unique). Then

$$\begin{aligned} \mathbf{P}\mathbf{x} \cdot \mathbf{y} &= \mathbf{P}(\mathbf{s} + \mathbf{t}) \cdot (\mathbf{s}' + \mathbf{t}') \\ &= \mathbf{s} \cdot (\mathbf{s}' + \mathbf{t}') \text{ since } \mathbf{P}\mathbf{x} = \mathbf{P}(\mathbf{s} + \mathbf{t}) = \mathbf{s} \\ &= \mathbf{s} \cdot \mathbf{s}' + \mathbf{s} \cdot \mathbf{t}' \\ &= \mathbf{s} \cdot \mathbf{s}' \text{ since } \mathbf{s} \cdot \mathbf{t}' = 0 \text{ because } \mathbf{t}' \in S^\perp. \end{aligned}$$

Similarly,

$$\mathbf{x} \cdot \mathbf{P}\mathbf{y} = (\mathbf{s} + \mathbf{t}) \cdot \mathbf{P}(\mathbf{s}' + \mathbf{t}') = (\mathbf{s} + \mathbf{t}) \cdot \mathbf{s}' = \mathbf{s} \cdot \mathbf{s}' + \mathbf{t} \cdot \mathbf{s}' = \mathbf{s} \cdot \mathbf{s}',$$

so that $\mathbf{P}\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{P}\mathbf{y}$. Since this holds for all $\mathbf{x}, \mathbf{y} \in X$, then $\mathbf{P}^T = \mathbf{P}$ and $\mathbf{P}$ is self adjoint, as claimed. □