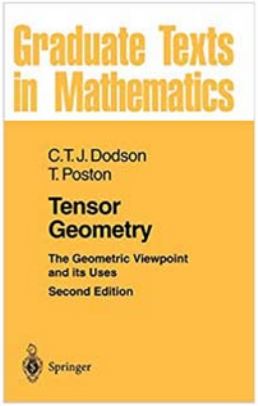
Differential Geometry

Chapter V. Tensors and Multilinear Forms

V.1. Multilinear Forms—Proofs of Theorems



Lemma V.1.05

Lemma V.1.05. A tensor product of finite dimensional vector spaces $X_1, X_2, \ldots, X_n$ always exists and any two are isomorphic "in a natural way."

Proof. We have already argued in this section that a tensor product of spaces $X_1^*, X_2^*, \ldots, X_n^*$ is given by $L(X_1, X_2, \ldots, X_n; \mathbb{R})$. By Note III.1.B, $X_i \cong (X_i^*)^*$ "naturally," so a tensor product of $X_1, X_2, \ldots, X_n$ is given by $L(X_1^*, X_2^*, \ldots, X_n^*; \mathbb{R})$, establishing existence.

For the isomorphism claim, suppose X and X' are tensor products with maps $\bigotimes$ and $\bigotimes'$, with properties (T i) and (T ii). Then by (T i), $\bigotimes : X_1 \times X_2 \times \cdots \times X_n \to X$ is multilinear and by (T ii) with $\mathbf{f} = \bigotimes$ and $Y = X$ there is a unique linear $\hat{\mathbf{f}} = \mathbf{\Psi} : X_1 \times X_2 \times \cdots \times X_n \to X$ such that $\bigotimes = \mathbf{\Psi} \circ \bigotimes'$. Similarly, by (T i), $\bigotimes' : X_1 \times X_2 \times \cdots \times X_n \to X'$ is multilinear and by (T ii) with $\mathbf{f} = \bigotimes'$ and $Y = X'$ there is a unique linear $\hat{\mathbf{f}} = \mathbf{\Phi} : X_1 \times X_2 \times \cdots \times X_n \to X'$ such that $\bigotimes' = \mathbf{\Phi} \circ \bigotimes$. Hence $\mathbf{\Psi} \circ \mathbf{\Phi} \circ \bigotimes = \mathbf{\Psi} \circ \bigotimes' = \bigotimes = \mathbf{I}_X \circ \bigotimes$.

Lemma V.1.05 (continued)

Lemma V.1.05. A tensor product of finite dimensional vector spaces $X_1, X_2, \ldots, X_n$ always exists and any two are isomorphic "in a natural way."

Proof (continued). One more time, by (T i) $\bigotimes = \mathbf{I}_X \circ \bigotimes$ is multilinear and by (T ii) with $\mathbf{f} = \bigotimes$ and $Y = X$ there is a unique linear function $\hat{\mathbf{f}}$ mapping $X_1 \times X_2 \times \cdots \times X_n \to X$ such that $\bigotimes = \hat{\mathbf{f}} \circ \bigotimes$. But as shown above we could take $\hat{\mathbf{f}} = \mathbf{I}_x$ or $\hat{\mathbf{f}} = \mathbf{\Psi} \circ \mathbf{\Psi}$, so we must have $\mathbf{\Psi} \circ \mathbf{\Psi} = \mathbf{I}_X$. Similarly (interchanging the roles of X and X' and of $\bigotimes$ and $\bigotimes'$ (and by the uniqueness of (T ii)) we have $\mathbf{\Psi} \circ \mathbf{\Psi} = \mathbf{I}_{X'}$. So $\mathbf{\Psi}$ and $\mathbf{\Psi}$ are inverses of each other and so are one to one and onto. So $\mathbf{\Psi}$ and $\mathbf{\Psi}$ are linear one to one and onto mappings between vector space X and X'. That is, $\mathbf{\Psi}$ and $\mathbf{\Psi}$ are vector space isomorphisms and $X \cong X'$ so that any two tensor products of $X_1, X_2, \ldots, X_n$ are isomorphic, as claimed. □

Lemma V.1.07

Lemma.1.07. There is a "natural" isomorphism yielding $X_1^* \otimes X_2^* \otimes \cdots \otimes X_n^* \cong (X_1 \otimes X_2 \otimes \cdots \otimes X_n)^*$.

Proof. Define

$$\mathbf{\Phi} : (X_1 \otimes X_2 \otimes \cdots \otimes_n)^* \to L(X_1, X_2, \ldots, X_n; \mathbb{R}) = X_1^* \otimes X_2^* \otimes \cdots \otimes X_n^*$$

as $\mathbf{\Phi}(\mathbf{f}) = \mathbf{f} \circ \bigotimes$ where $\bigotimes : X_1^* \times X_2^* \times \cdots \times X_n^* \to L(X_1, X_2, \ldots, X_n; \mathbb{R})$ is defined (as above) as $\bigotimes((\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n)) = \mathbf{g}_1 \otimes \mathbf{g}_2 \otimes \cdots \otimes \mathbf{g}_n$, and define $\mathbf{\Psi} : L(X_1, X_2, \ldots, X_n; \mathbb{R}) \to (X_1 \otimes X_2 \otimes \cdots \otimes X_n)^*$ as $\mathbf{\Psi}(\mathbf{g}) = \hat{\mathbf{g}}$ where $\hat{\mathbf{g}}$ is the unique function such that $\mathbf{g} = \hat{\mathbf{g}} \circ \bigotimes$ given in (T ii). Then $\mathbf{\Phi}$ is linear since

$$\mathbf{\Psi}(a\mathbf{f} + b\mathbf{f}') = (a\mathbf{f} + b\mathbf{f}') \circ \bigotimes = (a\mathbf{f}) \circ \bigotimes + (b\mathbf{f}') \circ \bigotimes$$

$$= a\left(\mathbf{f} \otimes \bigotimes\right) + b\left(\mathbf{f}' \circ \bigotimes\right) = a\mathbf{\Phi}(\mathbf{f}) + b\mathbf{\Phi}(\mathbf{f}').$$

Lemma V.1.07 (continued)

Lemma.1.07. There is a "natural" isomorphism yielding $X_1^* \otimes X_2^* \otimes \cdots \otimes X_n^* \cong (X_1 \otimes X_2 \otimes \cdots \otimes X_n)^*$.

Proof (continued). Now $\mathbf{\Phi} \circ \mathbf{\Phi}(\mathbf{f}) = \mathbf{\Psi}(\mathbf{f} \circ \bigotimes) = \mathbf{f}$ (take $\mathbf{g} = \mathbf{g} \circ \bigotimes$ and $\hat{\mathbf{g}} = \mathbf{f}$) and

$$\begin{aligned} \mathbf{\Phi} \circ \mathbf{\Phi}(\mathbf{f}) &= \mathbf{\Phi}(\hat{\mathbf{f}}) \text{ where } \mathbf{f} = \hat{\mathbf{f}} \circ \bigotimes \\ &= \hat{\mathbf{f}} \circ \bigotimes = \mathbf{f}, \end{aligned}$$

so $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are inverse functions and hence $\mathbf{\Phi}$ is a bijection and so is a vector space isomorphism (and so is $\mathbf{\Psi}$). So $X_1^* \otimes X_2^* \otimes \cdots \otimes X_n^* \cong (X_1 \otimes X_2 \otimes \cdots \otimes X_n)^*$, as claimed. □

Lemma V.1.08

Lemma V.1.08. For any two vector spaces X_1 and X_2, there is a "natural" isomorphism yielding $L(X_1; X_2) \cong X_1^* \otimes X_2$.

Proof. Define $\mathbf{f} : X_1^* \times X_2 \to L(X_1; X_2)$ as $\mathbf{f}((\mathbf{g}, \mathbf{x}_2)) = \mathbf{h}$ where $\mathbf{h}(\mathbf{x}_1) = \mathbf{x}_2(\mathbf{g}(\mathbf{x}_1))$ (notice $\mathbf{g}(\mathbf{x}_1) \in \mathbb{R}$). By Exercise V.1.7(a), $\mathbf{f}$ is multilinear. So by (T ii), $\mathbf{f}$ induces a unique linear map $\hat{\mathbf{f}} : X_1^* \otimes X_2 \to L(X_1; X_2)$ (here, $Y = L(X_1; X_2)$) such that $\mathbf{f} = \hat{\mathbf{f}} \circ \bigotimes$.

Now $\hat{\mathbf{f}}(\mathbf{g} \otimes \mathbf{x}_2) = \mathbf{0}$ implies $\mathbf{f}(\mathbf{g}_1, \mathbf{x}_2)) = \mathbf{0}$ since $\bigotimes((\mathbf{g}, \mathbf{x}_2)) = \mathbf{g} \otimes \mathbf{x}_2$. This implies that $\mathbf{f}((\mathbf{g}, \mathbf{x}_2)) = \mathbf{h} = \mathbf{0}$ and so $\mathbf{h}(\mathbf{x}_1) = \mathbf{0}(\mathbf{x}_1) = \mathbf{x}_2(\mathbf{g}(\mathbf{x}_1)) = \mathbf{0}$ for all $\mathbf{x}_1 \in X_1$. So either scalar $\mathbf{g}(\mathbf{x}_1) = 0$ for all $\mathbf{x}_1 \in X_1$ (i.e., $\mathbf{g} = \mathbf{0}$) or vector $\mathbf{x}_1 = \mathbf{0}$. This implies $\mathbf{g} \otimes \mathbf{x}_2 = \mathbf{0}$ (if $\mathbf{g} = \mathbf{0}$ then by (T S) for $a = 2$ we have $\mathbf{0} \otimes \mathbf{x}_2 = (2\mathbf{0}) \otimes \mathbf{x}_2 = 2(\mathbf{0} \otimes \mathbf{x}_2)$ and by (T A) $\mathbf{0} \otimes \mathbf{x}_2 = \mathbf{0}$, and similarly for $\mathbf{x}_2 = \mathbf{0}$). That is, $\mathbf{f}(\mathbf{g} \otimes \mathbf{x}_2) = \mathbf{0}$ implies $\mathbf{g} \otimes \mathbf{x}_2 = \mathbf{0}$. By Exercise V.1.7(b), this implies that $\hat{\mathbf{f}}$ is one to one (injective).

Lemma V.1.08 (continued)

Lemma V.1.08. For any two vector spaces X_1 and X_2, there is a "natural" isomorphism yielding $L(X_1; X_2) \cong X_1^* \otimes X_2$.

Proof (continued). Next, since X_1 and X_1^* are finite dimensional then

$$\begin{aligned} \dim(X_1^* \otimes X_2) &= \dim(X_1^*)\dim(X_2) \text{ by Exercise V.1.4(c)} \\ &= \dim(X_1)\dim(X_2) \text{ by Lemma III.1.04} \\ &= \dim(L(X;Y)) \text{ (see page 27 or think matrices).} \end{aligned}$$

So $\hat{\mathbf{f}}$ is onto (surjective), and hence $\hat{\mathbf{f}}$ is a vector space isomorphism and $L(X_1; X_2) \cong X_1^* \otimes X_2$, as claimed. □

Lemma V.1.B

Lemma V.1.B. Let $\mathbf{A}_i : X_i \to Y_i$ be linear maps for $1 \le i \le n$. Mapping $\mathbf{h} : X_1 \times X_2 \times \cdots \times X_n \to Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n$ defined as $\mathbf{h} = \bigotimes \circ (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n)$ is multilinear.

Proof. Let $\mathbf{x}_i, \mathbf{x}' \in X_i$ for $1 \le i \le n$ and let $a \in \mathbb{R}$. Then

$$\begin{aligned} &\mathbf{h}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_i + \mathbf{x}_i', \ldots, \mathbf{x}_n) \\ = &\bigotimes((\mathbf{A}_1\mathbf{x} - 2, \mathbf{A}_2\mathbf{x}_2, \ldots, \mathbf{A}_i(\mathbf{x}_i + \mathbf{x}_i'), \ldots, \mathbf{A}_n\mathbf{x}_n) \\ &\text{by definition of } \mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \cdots \otimes \mathbf{A}_n \\ = &\bigotimes((\mathbf{A}_1\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}_i\mathbf{x}_i + \mathbf{A}_i\mathbf{x}_i', \ldots, \mathbf{A}_n\mathbf{x}_n)) \text{ since } \mathbf{A}_i \text{ is linear} \\ = &\bigotimes((\mathbf{A}_1\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}_i\mathbf{x}_i, \ldots, \mathbf{A}_n\mathbf{x}_n) \\ &+(\mathbf{A}_1\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}_i\mathbf{x}_i', \ldots, \mathbf{A}_n\mathbf{x}_n)) \text{ by the definition} \\ &\text{of vector addition in } Y_1 \times Y_2 \times \cdots \times Y_n \ldots \end{aligned}$$

Lemma V.1.B (continued)

Proof (continued).

$$= \bigotimes((\mathbf{A}_1\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}_i\mathbf{x}_i, \ldots, \mathbf{A}_n\mathbf{x}_n))$$
$$+ \bigotimes((\mathbf{A}_1\mathbf{x}_1, \mathbf{A}\mathbf{x}_2, \ldots, \mathbf{A}_i\mathbf{x}_i', \ldots, \mathbf{A}_n\mathbf{x}_n))$$
since $\bigotimes$ is linear (see Definition V.1.04)

and

$$\mathbf{h}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_i a, \ldots, \mathbf{x}_n)$$
$$= \bigotimes((\mathbf{A}_1\mathbf{x}_1, \mathbf{A}_2\mathbf{x}_2, \ldots, \mathbf{A}_i(\mathbf{x}_i a), \ldots, \mathbf{A}_n\mathbf{x}_n))$$
$$= \bigotimes((\mathbf{A}_1\mathbf{x}_1, \mathbf{A}_2\mathbf{x}_2, \ldots, (\mathbf{A}_i\mathbf{x}_i)a, \ldots, \mathbf{A}_n\mathbf{x}_n) \text{ since } \mathbf{A}_i \text{ is linear}$$
$$= \bigotimes((\mathbf{A}_1\mathbf{x}_1, \mathbf{A}_2\mathbf{x}_2, \ldots, \mathbf{A}_i\mathbf{x}_i, \ldots, \mathbf{A}_n\mathbf{x}_n))a$$
since $\bigotimes$ is linear (see Definition V.1.04)
$$= \mathbf{h}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)a.$$

Hence **h** is multilinear, as claimed. □