Differential Geometry (and Relativity) - Summer 2019

Homework 1, Section 1.1

Due Friday, June 7 at 1:00

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why.

I.1.6(a) Show that the plane curve $\vec{\alpha}(t) = (x(t), y(t))$ has curvature

$$k(t) = \left| \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}} \right|$$

at $\vec{\alpha}(t)$. Notice from Exercise I.1.5 that we have $k(t) = \|d\vec{T}/dt\|/\|\vec{\alpha}'(t)\|$ and $\vec{T} = \vec{\alpha}'(t)/\|\vec{\alpha}'(t)\|$.

I.1.6(b) As a special case, show that the graph of $y = f(x)$ has curvature

$$k(x) = \left| \frac{f''(x)}{(1 + f'(x)^2)^{3/2}} \right|$$

at $(x, f(x))$.

I.1.7(a) Let $\vec{\alpha}(t) = (a\cos t, b\sin t)$ for $0 \le y \le 2\pi$. Since $x^2/a^2 + y^2/b^2 = 1$, the image of $\vec{\alpha}$ is an ellipse. Compute its curvature $k(t)$ by the formula of Exercise I.1.6(a) at $t = 0$ and $t = \pi/2$.

I.1.7(b) Sketch the ellipse $x^2/4 + y^2 = 1$ and its osculating circles at the points $(2, 0)$ and $(0, 1)$. Find the equations of the osculating circles.

I.1.9. Let $\vec{\alpha}(t)$ be a smooth curve in E^3, where t is an arbitrary parameter. Let $v(t) = ds/dt$ be the speed at parameter value t. Then

$$\vec{\alpha}'(t) = \frac{d\vec{\alpha}}{ds}\frac{ds}{dt} = v\vec{T} \text{ and } \vec{T}'(t) = \frac{d\vec{T}}{ds}\frac{ds}{dt} = kv\vec{N}.$$

Prove that

$$k = \frac{\|\vec{\alpha}' \times \vec{\alpha}''\|}{\|\vec{\alpha}'\|^3}$$

where the primes indicate derivatives with respect to t.

I.1.10. Compute the curvature of the helix $\vec{\beta}(t) = (a\cos t, a\sin t, bt)$ (of Example 3) by means of the formula derived in Exercise I.1.9.

I.1.11(c) Compute the curvature $k(t)$ for $\vec{\alpha}(t) = (\cos t, \sin t, e^t)$.