Chapter II. Affine Spaces

II.1. Spaces

Note. In this section, we define an affine space on a set X of points and a vector space T. In particular, we use affine spaces to define a tangent space to X at point x. In Section VII.2 we define manifolds on affine spaces by mapping open sets of the manifold (taken as a Hausdorff topological space) into the affine space. Ultimately, we think of a manifold as pieces of the affine space which are "bent and pasted together" (like paper mache). We also consider subspaces of an affine space and translations of subspaces. We conclude with a discussion of coordinates and "charts" on an affine space.

Definition II.1.01. An *affine space* with vector space T is a nonempty set X of *points* and a vector valued map $\mathbf{d} : X \times X \to T$ called a *difference function*, such that for all $x, y, z \in X$:

(A i) $\mathbf{d}(x, y) + \mathbf{d}(y, z) = \mathbf{d}(x, z)$,

(A ii) the restricted map $\mathbf{d}_x = \mathbf{d}|_{\{x\}\times X} : \{x\} \times X \to T$ defined as mapping $(x, y) \mapsto \mathbf{d}(x, y)$ is a bijection.

We denote both the set and the affine space as X.

Note. We want to think of a vector as an arrow between two points (the "head" and "tail" of the vector). So $\mathbf{d}(x, y)$ is the vector from point x to point y. Then we see that (A i) is just the usual "parallelogram property" of the addition of vectors. Property (A ii) can be thought of in terms of putting vectors in a "standard position" with their tails all at point x. In fact, we show below in Lemma II.1.A that $\mathbf{d}(x, x) = \mathbf{0} \in T$ and so it could be convenient to thank point x as the origin for all vectors with their tails at x. We will see in Exercise II.1.1 that all such vectors in T of the form $\mathbf{d}(x, y)$ form a vector space. Notice that an affine space consists of a set X, vector space T, and difference function $\mathbf{d}$; however, an affine space (per se) is not itself a vector space.

Examples. With $X = \mathbb{R}$ (as a point set) and $T = \mathbb{R}^1$ (as a vector space), we can define $\mathbf{d}(x, y) = y - x$. We can similarly let $X = \mathbf{R}^n$ (as a point set), $T = \mathbb{R}^n$ (as a vector space) and define

$$\mathbf{d}((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = (y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n).$$

These are examples of affine spaces where $\mathbf{d}(x, y)$ is the vector in $\mathbb{R}^n$ from point x to point y (with the usual arrow with magnitude-an-direction interpretation, valid in $\mathbb{R}^n$). Notice that in these examples, (A i) simply expresses the summation of vectors in $\mathbb{R}^n$ and (A ii) sets up a bijection where $\mathbf{x}$ is mapped to the $\mathbf{0}$ vector (as we now show).

Lemma II.1.A. In an affine space with difference function $\mathbf{d}$ we have

(a) $\mathbf{d}(\mathbf{x}, \mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in X$, and

(b) $\mathbf{d}(\mathbf{x}, \mathbf{y}) = -\mathbf{d}(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in X$.

Note II.1.A. Notationally, in an affine space if $\mathbf{d}(x, y) = \mathbf{t}$ then we write $y = x + \mathbf{t}$; think that we start at point x and go "directed distance" $\mathbf{t}$ ending at point y. If $V \subseteq T$ then we write $x + V = \{\mathbf{d}(x, y) = \mathbf{t} \mid \mathbf{t} \in V\} = \{x + \mathbf{t} \mid \mathbf{t} \in V\}$.

Note/Definition II.1.02. In an affine space, define for $x, y, z \in X$ and $a \in \mathbb{R}$

$$(x, y) + (x, z) = \mathbf{d}_x^{-1}(\mathbf{d}_x(x, y) + \mathbf{d}_x(x, z))$$

$$(x, y)a = \mathbf{d}_x^{-1}((\mathbf{d}(x, y)a)).$$

Then by Exercise II.1.1(a), these two operations define vector addition and scalar multiplication so that $\{x\} \times X$ has a vector space structure. This vector space is the *tangent space* to X at point x, denoted $T_x X$. For $\mathbf{v} \in T_x X$, we denote $x + \mathbf{d}_x^{-1}(\mathbf{v})$ as $x + \mathbf{v}$.

Note. Since $T_x X \cong T$ for each $x \in X$ by Exercise II.1.1(b), we see that $V = \{(x, y) \mid x, y \in X\}$ can be interpreted as a vector space with a few added details. We could define an equivalence relation on V where $(x, y) \sim (w, z)$ if and only if in T $\mathbf{d}(x, y) = \mathbf{d}(w, z)$ and then consider $W = V_\sim$, the set of equivalence classes in V. Then we could define vector addition and scalar multiplication with Definition II.1.02 (where x can vary) on equivalence classes in terms of representation (we would need to check that the definitions are well-defined). In addition, since $\mathbf{d}_x$ is a bijection between $T_x X$ and T then we see that $\mathbf{d}(X \times X)$ is trivially a vector space since $\mathbf{d}(X \times X) = T$.

Definition. In an affine space with tangent space at x of T_xX, the vectors in T_xX are called *tangent* (or *bound*) *vectors at* x. The vectors in vector space T are *free vectors*. Mapping $\mathbf{d}_x : \{x\} \times X \to T$ is a *freeing map* and $\mathbf{d}_x^{-1} : T \to \{x\} \times X$ is a *binding map*.

Definition. Let X be an affine space with vector space T and difference function $\mathbf{d}$. Then $X' \subseteq X$ is an *affine subspace* (or *flat*) of X if

(i) $\mathbf{d}(X', X') \subseteq T$ is a vector subspace of T, and

(ii) X' is an affine space with vector space $\mathbf{d}(X', X')$ and difference function $\mathbf{d}|_{X' \times X'} : X' \times X' \to \mathbf{d}(X', X')$ (so that for $(x, y) \in X' \times X'$ we have $\mathbf{d}|_{X' \times X'}(x, y) = \mathbf{d}(x, y)$.

Recall that a *hyperplane* of a n-dimensional vector space is an $(n-1)$-dimensional subspace. If $\mathbf{d}(X' \times X')$ is a hyperplane of T, then X' is an *affine hyperplane* of X.

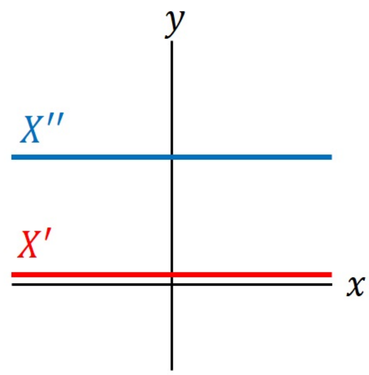
Note/Definition. As observed in the examples above for $\mathbb{R}^n$, we could take any vector space X and define $\mathbf{d}(X, X) \to X$ as $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{y} - \mathbf{x}$, giving an affine space with vector space X and difference function $\mathbf{d}$. This is called the *natural affine structure* on vector space X.

Note. An intersection of subspaces of a vector space is again a subspace. This gives us one way to define an affine subspace generated by a set S.

Definition. Let X be an affine space with vector space T and let $S \subseteq X$. The *affine subspace generated* by set S (or *affine hull*) $H(S)$ of S is the intersection of all affine subspaces of X containing S. This is the "smallest" (in a set inclusion sense) affine subspace of X containing S.

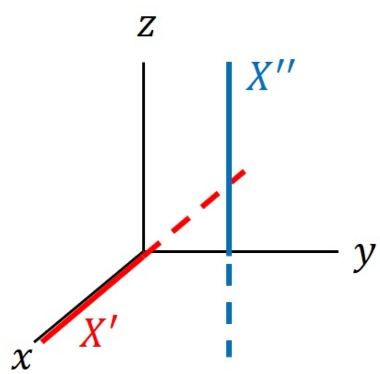
Definition II.1.04. The *translate* $X' + \mathbf{t}$ of an affine subspace X' of X by a vector $\mathbf{t} \in T$ is the affine subspace $\{y = x' + \mathbf{t} \mid x' \in X'\}$. Two affine subspaces X' and X'' of X are *parallel* if $\mathbf{d}(X' \times X') = \mathbf{d}(X'' \times X'')$.

Note. In $\mathbb{R}^n$, we might think of parallel structures as "flat" slices of $\mathbb{R}^n$ that do not intersect. For example, in $\mathbb{R}^3$ two planes are parallel if they are translates of each other. However, two line in $\mathbb{R}^3$ can be parallel (that is, not intersect) without being translates of each other. Additionally, if we consider subspace of $\mathbb{R}^n$ they must intersect since every subspace of $\mathbb{R}^n$ must contain $\mathbf{0}$. In the next lemma we see that subspaces X' and X'' of affine space X are parallel if and only if $X'' = X' + \mathbf{t}$ for some $\mathbf{t} \in T$. This means that "parallel" in the affine space setting has more of a "same distance apart" interpretation then the more traditional "not intersecting" interpretation from geometry. For example, with $X = \mathbb{R}^2$, $X' = \{(x, 0) \mid x \in \mathbb{R}\}$, $X'' = \{(x, 1) \mid x \in \mathbb{R}\}$, $T = \mathbb{R}^2$, and $\mathbf{d}((x_1, y_1), (x_2, y_2)) = (x_2 - x_1)\imath + (y_2 - y_1)\jmath$, we have $\mathbf{d}(X' \times X') = \mathbf{d}(X'' \times X'') = \{x\imath \mid x \in \mathbb{R}\} \subseteq T$. So X' and X'' are parallel. Notice in $X = \mathbf{R}^2$ that we have:



In Definition II.1.07 below, we'll define $\dim(X') = \dim((X' \times X'))$ and $\dim(X'') = \dim(\mathbf{d}(X'' \times X''))$, so in fact our definition of parallel affine subspaces requires that the subspaces be of the same dimension. However, $\dim(X') = \dim(X'')$ is not sufficient for our version of parallel, as we now show by example.

Example. Let $X = \mathbb{R}^3$, $X' = \{(x, 0, 0) \mid x \in \mathbb{R}\}$, $X'' = \{(0, 1, z) \mid z \in \mathbb{R}\}$, $T = \mathbb{R}^3$, and $\mathbf{d}((x_1, y_1, z_1)) = (x_2 - x_1)\imath + (y_2 0 y_1)\jmath + (z_2 - z_1)\mathbf{k}$. Then $\mathbf{d}(X' \times X') = \{x\imath \mid x \in \mathbb{R}\}$ and $\mathbf{d}(X'' \times X'') = \{z\mathbf{k} \mid z \in \mathbb{R}\}$. So subspace X' and X'' are not parallel, though $\dim(X') = \dim(X'') = 1$. Notice that in $\mathbb{R}^3$, sets X' and X'' are skew lines:



Note. By Exercise II.1.2(c), $X' + \mathbf{t}$ is itself an affine subspace of X. By Exercise II.1.2(b), $X' + \mathbf{t} = \{(x' + \mathbf{t}) + \mathbf{s} \mid \mathbf{s} \in \mathbf{d}(X' \times X')\}$ for every $x' \in X'$.

Lemma II.1.05. Two affine subspaces X' and X'' of X are parallel if and only if $X'' = X' + \mathbf{t}$ for some $\mathbf{t} \in T$.

Note. In light of Lemma II.1.05, it might make more sense to *define* affine subspaces X' and x'' as parallel if $X'' = X' + \mathbf{t}$ for some $\mathbf{t} \in T$. The concept of parallel from geometry carries with it an idea of non-intersection so we might want to add the constraint that $\mathbf{t} \in T$ but $\mathbf{t} \not\in \mathbf{d}(X' \times X')$; for $\mathbf{t} \in \mathbf{d}(X' \times X')$ then $X'' = X'$.

Lemma II.1.06. For X a vector space, $X' \subseteq X$ is an affine subspace of X if and only if X' is a translate of some vector subspace of X.

Definition II.1.07. Let X be an affine space with vector space T. The *dimension* of affine space X is the dimensional of its space T of free vectors; i.e., $\dim(X) = \dim(T)$.

Note. Let X be an affine space with vector space T (of dimension n) and difference function $\mathbf{d}$. Since $\dim(T) = n$ then by the Fundamental Theorem of Finite Dimensional Vector Spaces (see Theorem 3.3.A of "3.3. Coordinatization of Vectors" in my online notes for Linear Algebra [MATH 2010]) we have $T \cong \mathbb{R}^n$. For a given $a \in X$ (which will act as an "origin" for a coordinate system), we know that as a vector space (defined in Definition II.1.02) $T_aX \cong T$ by Exercise II.1.1(b). So for a given $a \in X$ we have a bijection mapping $T_aX \mapsto T \mapsto \mathbb{R}^n$. Let X_a denote this bijection. Notice $X_1 = \mathbf{a} \circ \mathbf{d}_a$ where $\mathbf{A} : T \to \mathbb{R}^n$ is a vector space isomorphism.

Definition. Let X be an affine space with vector space T and difference function $\mathbf{d}$. Let $a \in X$ and let $C_a : T_a X \to \mathbb{R}^n$ be a bijection as described above. Then C_a is a *chart* (or a *choice of coordinates*) on X.

Note. This is similar to our approach later when we define charts on manifolds modeled on affine spaces.

Note. In the chart C_x above, a basis $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ for T defines a basis $\beta_x = \{\mathbf{d}_x^{-1}(\mathbf{b}_1), \mathbf{d}_x^{-1}(\mathbf{b}_2), \ldots, \mathbf{d}_x^{-1}(\mathbf{b}_n)\} = \{\beta_{1x}, \beta_{2x}, \ldots, \beta_{nx}\}$ of $T_x X$. If we make two choices of the origin, a and a', and two choices of basis, β and β', then to change from the first coordinate system (unprimed) to the second coordinate system (primed) we must map $\mathbb{R}^n$ to $\mathbb{R}^n$ by first converting from basis β to β' with matrix $[\mathbf{I}_\beta^{\beta'}$ and then translating "origin" a' to "origin" a by adding $\mathbf{A}(\mathbf{d}(a', a))$ (i.e., the vector from a' to a in the β' system):

$$\begin{bmatrix} x^{1\,\prime} \\ x^{2\,\prime} \\ \vdots \\ x^{n\,\prime} \end{bmatrix} = [\mathbf{I}]_\beta^{\beta'} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \mathbf{A}(\mathbf{d}(a', a))$$

or $x^{i\,\prime} = b_j^i x^j + a^i$ where b_j^i is the ith coordinate in the β' system of the jth vector in β, and a^i is the ith coordinate of the vector $\mathbf{d}(a', a)$ in the β' system.

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