II.2. Combinations of Points

Note. We define an affine combination of points in an affine space.

Note. If x and y are points in affine space X (with vector space T and difference function $\mathbf{d}$) then we can describe the "midpoint" of x and y as $x + \frac{1}{2}\mathbf{d}(x, y)$ or $y + \frac{1}{2}(y, x)$. Notice that this is meaningful, for if $z = x + \frac{1}{2}\mathbf{d}(x, y)$ then $\mathbf{d}(x, z) = \frac{1}{2}\mathbf{d}(x, y)$. Since $\mathbf{d}(x, y) \in T$ then $\frac{1}{2}\mathbf{d}(x, y)$ is meaningful and since $\mathbf{d}_x : \{x\} \times X \to T$ is a bijection then point z is uniquely determined from the equation $\mathbf{d}(x, z) = \frac{1}{2}\mathbf{d}(x, y)$. More generally, we can define any point on a line joining x and y.

Definition II.2.01. Let X be an affine space with vector space T and difference function $\mathbf{d}$. For any $\mu, \lambda \in \mathbb{R}$ with $\mu + \lambda = 1$ and for any $x, y \in X$, the *affine combination* $\mu x + \lambda y$ is the point in X defined equivalently by $x + \lambda\mathbf{d}(x, y)$ or $y + \mu\mathbf{d}(y, x)$.

Note. By taking repeated affine combinations of points two at a time, we can deal with a more general affine combination.

Definition. Let X be an affine space with vector space T and difference function $\mathbf{d}$. For any $x_1, x_2, \ldots, x_k \in X$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ such that $\sum_{i=1}^{k} \lambda_i = 1$, the *affine combination* $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ is defined as

$$\lambda_1 x_1 + (1-\lambda_1)\left(\frac{\lambda_2}{1-\lambda_1}x_2 + \left(1 - \frac{\lambda_2}{1-\lambda_1}\right)\left(\frac{\lambda_3}{1-\lambda_1-\lambda_2}x_3\right.\right.$$

$$\left.\left.+\left(1-\frac{\lambda_3}{1-\lambda_1-\lambda_2}\right)\left(\frac{\lambda_4}{1-\lambda_1-\lambda_2-\lambda_3}x_4 + \left(\cdots + \left(1 - \frac{\lambda_{n-1}}{1-\lambda_1-\lambda_2-\cdots-\lambda_{n-2}}\right)x_n\right)\cdots\right)\right.\right).$$

Note. In Exercise II.2.1(b), it is shown that the order of the points (and λ's) can be permuted around and the formula for the affine combination remains unchanged.

Note. In an affine space X, the affine hull $H(S)$ for $S \subseteq X$ satisfies

$$H(S) = \left\{ \lambda_1 x_2 + \lambda_2 x_2 + \cdots + \lambda_k x_k \;\middle|\; x_i \in S \text{ for } i \in \{1,2,\ldots,k\}, \sum_{i=1}^{k} \lambda_i = 1, k \in \mathbb{N} \right\},$$

as is to be shown in Exercise II.2.2(a). Set $S \subseteq X$ is an affine subspace of X if and only if $H(S) = S$ by Exercise II.2.2(b).

Definition. Let X be an affine space and $S \subseteq X$. The *convex hull* $C(S)$ is the set

$$\left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \;\middle|\; x_i \in S, \lambda_i > 0, i \in \{1,2,\ldots,k\}, \sum_{i=1}^{k} \lambda_i = 1, k \in \mathbb{N} \right\}.$$

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