II.3. Maps

Note. Following the definition of an affine combination of points in an affine space in the previous section, we now define an affine map between affine spaces and address some of their properties.

Definition II.3.01. A map $A : (X, T) \to (Y, S)$ between affine spaces (where X and Y are the point sets and T and S are the vector spaces) is an *affine map* if for all $x, x' \in X$ and for all $\lambda \in \mathbb{R}$ we have

$$A((a-\lambda)x + \lambda x') = (a-\lambda)Ax + \lambda Ax'. \qquad (*)$$

If $P \subseteq X$ and $Q \subseteq Y$ are convex sets, then $A : P \to Q$ is an *affine map on convex sets* of all $x, x' \in P$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$, $(*)$ holds.

Note. By the recursive definition of affine combination in the previous section we have for affine map A with $a_1, x_2, \ldots, x_k \in X$, and $\sum_{i=1}^k \lambda_i = 1$ that

$$A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) = \lambda_1 Ax_1 + \lambda_2 Ax_2 + \cdots + \lambda_k Ax_k.$$

Exercise II.3.2(a) can be used to give the following relation between affine combination and affine hulls for set $S \subseteq X$: $A(H(S)) = H(A(S))$. In addition, Exercise II.2.2(b) can be used to show that an affine map carries flats to flats (i.e., affine subspaces to affine subspaces; see Definition II.1.03).

Note/Definition II.3.02. If $A : (X, T) \to (Y, S)$ is an affine map then, as shown in Exercise II.3.3(a), there is a corresponding map $\mathbf{A} : T \to S$ which is (by Exercise II.3.3(b) and 3(c)) a linear map between vector spaces T and S. The map $\mathbf{A}$ is given by

$$\mathbf{A} = \{(\mathbf{t}, \mathbf{s}) \mid \text{ there exists } x \in X \text{ such that } \mathbf{d}(Ax, A(x + \mathbf{t})) = \mathbf{s}\}.$$

That is, $\mathbf{t} = \mathbf{s}$ where $\mathbf{s}$ is as described. Vector space mapping $\mathbf{A}$ is the *linear part* of affine space mapping A.

Note. Let $A : (X, T) \to (Y, S)$ be an affine map. For $\mathbf{t} \in T$ we have $\mathbf{At} = \mathbf{s}$ where $\mathbf{d}_2(Ax, A(x + \mathbf{t})) = \mathbf{s}$ for some $x \in X$ (where $\mathbf{d}_2$ is the difference function on X); that is, $A(x + \mathbf{t}) = Ax + \mathbf{s}$. So the pair of points x and $x + \mathbf{t}$ in X are mapped by A to the pair of points Ax and $A(x + \mathbf{t})$ in Y; that is, points in X "separated by" free vector $\mathbf{t} \in T$ are mapped by A to points in Y "separated by" free vector $\mathbf{s} \in S$. So we denote $\mathbf{s} = \mathbf{At}$. For any $x_0 \in X$ we have

$$A(x) = A(x_0 + \mathbf{d}(x_0, x)) = Ax_0 + \mathbf{A}(\mathbf{d}(x_0, x)) \qquad (*)$$

(here, $x = x_0 + \mathbf{d}(x_0, x)$ and $\mathbf{s} = \mathbf{A}(\mathbf{d}(x_0, x))$).

Definition II.3.03. An affine map $A : X \to Y$ is an *affine isomorphism* if there is an affine map $B :\to X$ such that AB and BA are identity maps. As affine isomorphism $X \to X$ is an *affine automorphism*. A *translation* of X is a map of the form $x \mapsto x + \mathbf{t}$ for some $\mathbf{t} \in T$.

Note. A translation is an affine transformation (since the map $x \mapsto x + (-\mathbf{t})$ is its inverse).

Definition II.3.04. The image AX of an affine map $A : X \to Y$ is its set-theoretic image $\{Ax \mid x \in X\}$. Since X is (trivially) an affine subspace of itself, AX is a flat of Y. The *rank* of A is $r(A) = \dim(AX)$. The *nullity* of A, $n(A)$, is the nullity of the linear part of A, $n(\mathbf{A})$.

Note. If we choose origins $\mathbf{0}_X$, $\mathbf{0}_Y$ for X and Y and bases for T and S, then for affine map A we have from $(*)$ that, in terms of components,

$$(Ax)^i = A^i_j x^j + a^i,\ 1 \le j \le n,\ 1 \le i \le m$$

where $[A^i_j]$ is the matrix representation of $\mathbf{A}$ in terms of the bases for T and S, $(a^1, a^2, \ldots, a^m)$ are the coordinates of $A(\mathbf{0}_X) \in Y$ in the cart used on Y, $(x^1, x^2, \ldots, x^n)$ are the coordinates of $x \in X$ in the chart used on X, and $((Ax)^1,$ $(Ax)^2, \ldots, (Ax)^m)$ are the coordinates of $Ax \in Y$ in the chart used on Y. Here, X is n-dimensional and Y is m-dimensional.

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