Chapter IV. Metric Vector Spaces

IV.1. Metrics

Note. In this section we consider bilinear forms, inner products, and norms. Dodson and Poston discuss dot products and angles between vectors in $\mathbb{R}^2$. We start with bilinear forms.

Definition IV.1.01. A *bilinear form* on a vector space X is a function $\mathbf{F} : X \times X \to \mathbb{R}$ that is linear in each variable as follows. For all $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in X$ and for all $a \in \mathbb{R}$:

(B i) $\mathbf{F}(\mathbf{x} + \mathbf{x}', \mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}) + \mathbf{F}(\mathbf{x}', \mathbf{y})$ and $\mathbf{F}(\mathbf{x}, \mathbf{y} + \mathbf{y}') = \mathbf{F}(\mathbf{x}, \mathbf{y}) + \mathbf{F}(\mathbf{x}, \mathbf{y}')$.

(B ii) $\mathbf{F}(\mathbf{x}a, \mathbf{y}) = a\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}a)$ (we follow Dodson and Poston's notation here of putting scalars on the right of vectors).

A bilinear form in X is

(i) *symmetric* if $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{F}(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in X$,

(ii) *anti-symmetric* if $\mathbf{F}(\mathbf{x}, \mathbf{y}) = -\mathbf{F}(\mathbf{x}, \mathbf{y})$,

(iii) *non-degenerate* if $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ for all $\mathbf{y} \in X$ implies $\mathbf{x} = \mathbf{0}$,

(iv) *positive definite* if $\mathbf{F}(\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,

(v) *negative definite* if $\mathbf{F}(\mathbf{x}, \mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$,

(vi) *indefinite* if it is neither positive definite nor negative definite.

Also,

(vii) A *metric tensor* on X is a symmetric non-degenerate bilinear form.

(viii) An *inner product* on X is a positive or negative definite metric tensor. We will always take **inner products to be positive definite**! (The "geometry" is the same whether the inner product is positive definite or negative definite since orthogonality and the angles between vectors are the same in both cases.)

(ix) A *symplectic structure* on X is a anti-symmetric non-degenerate bilinear form.

Note. Our text book does not address symplectic structures, but Dodson and Poston claim that they play a central role in classical mechanics.

Note. In Exercise IV.1.4, it is to be shown that the set of all bilinear forms on X form a real vector space. We denote this vector space as $L^2(X, \mathbb{R}) = L(X, X; \mathbb{R})$.

Note. An inner product is, by definition, a metric tensor. A metric tensor is not required to be positive definite. The Lorentz metric tensor (which will will somewhat misleadingly call the "Lorentz metric") from special relativity will be an example of an indefinite metric tensor.

Definition IV.1.02. A *metric vector space* $(X, \mathbf{G})$ is a vector space X with a metric tensor $\mathbf{G} : X \times X \to \mathbb{R}$. An *inner product space* $(X, \mathbf{G})$ is a vector space X with an inner product $\mathbf{G} : X \times X \to \mathbb{R}$.

Note. In a metric vector space $(X, \mathbf{G})$ we will often abbreviate $\mathbf{G}(\mathbf{x}, \mathbf{y})$ as $\mathbf{x} \cdot \mathbf{y}$ (even though $\mathbf{G}$ may not be positive definite or negative definite and so may not be an inner product), and refer to the metric vector space simply as X. We will use the symbol $\mathbf{G}$ exclusively for metric tensors (including inner products).

Definition IV.1.03. The *standard inner product* on $\mathbb{R}$ is defined by

$$(x^1, x^2, \ldots, x^n) \cdot (y^1, y^2, \ldots, y^n) = x^1 y^1 + x^2 y^2 + \cdots + x^n y^n = \sum_{i=1}^{n} x^i y^i.$$

Note. We know from the properties of dot products on $\mathbb{R}^n$ from Linear Algebra (MATH 2010) that it is symmetric $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$, non-degenerate $\mathbf{x} \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in \mathbb{R}^n$ implies $\mathbf{x} = \mathbf{0}$, and a bilinear form (i.e., linear in each variable). See "Theorem 1.3. Properties of Dot Products" in my online Linear Algebra notes for 1.2. The Norm and Dot Product.

Definition. The *Lorentz metric* on $\mathbb{R}^4$ is defined as the inner product

$$\mathbf{x} \cdot \mathbf{y} = (x^0, x^1, x^2, x^3) \cdot (y^0, y^1, y^2, y^3) = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3.$$

Notice that this gives $\mathbf{x} \cdot \mathbf{x} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$. The geometry of $\mathbb{R}^4$ with the Lorentz metric is explored in "IX. 6. An Example of Lie Group Geometry."

Definition. The *determinant metric* on $\mathbb{R}^4$ is defined as the inner product

$$\mathbf{x} \cdot \mathbf{y} = (x^1, x^2, x^3, x^4) \cdot (y^1, y^2, y^3, y^4) = \frac{1}{2}(x^1 y^4 + x^4 y^1) - \frac{1}{2}(x^3 y^2 + x^2 y^3).$$

Notice that this gives

$$\mathbf{x} \cdot \mathbf{x} = \det \begin{bmatrix} x^1 & x^2 \\ x^3 & x^4 \end{bmatrix} = x^1 x^4 - x^2 x^3.$$

Definition IV.1.04. In a vector space X with metric tensor $\mathbf{G}$, the *length* of $\mathbf{x} \in X$ is $|\mathbf{x}|_{\mathbf{G}} = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

Note. If metric $\mathbf{G}$ is not positive definite, then $|\mathbf{x}|_{\mathbf{G}}$ will be complex. Then we need to explore branches of the square root function (just as we do in the real setting where "$\sqrt{x}$" denotes the positive square root when $x > 0$). Since we will only consider real valued inner products, we only need to agree on the value of $\sqrt{x}$ for $x < 0$. We take this as $i\sqrt{|x|}$.

Note/Definition. With the Lorentz metric (tensor), we have the lengths

$$|(1,0,0,0)| = 1, \ |(1,1,0,0,0)| = |(1,0,1,0,0)| = |(1,0,0,1)| = 0,$$

$$|(0,1,0,0)| = |(0,0,1,0)| = |(0,0,0,1)| = \sqrt{-1} = i.$$

In special relativity, we consider spacetime as a collection of points (as opposed to vectors) called *events* and use the Lorentz metric tensor to measure distances between two points using "proper time τ" as

$$(\Delta\tau)^2 = (x^0 - y^0)^2 - (x^1 - y^1)^2 - (x^2 - y^2)^2 - (x^3 - y^3)^2.$$

The quantity $(\Delta\tau)^2$ is called the *interval* associated with events (x^0, x^1, x^2, x^3) and (y^0, y^1, y^2, y^3). If $(\Delta\tau)^2 > 0$ then the events are separated by more time than space and the interval is *timelike*. If $(\Delta\tau)^2 < 0$ then the events are separated by more space than time and the interval is *spacelike*. If $\Delta\tau = 0$ then the events are equally separated in time and in space and the interval is *lightlike*. An inertial observer (that is, an observer that is not accelerating) can be present at two events separated by a timelike interval; there is enough time separation to travel over the time separation. A photon traveling at the speed of light can be present at two events separated by a lightlike interval (but an observer cannot do this since by one of the Principles of Relativity, an observer must have a velocity less than that of light). Neither an inertial observer nor a photon can be present at two events separated by a spacelike interval; there is too much space separation to travel over during the amount of time in the time separation. See my online notes for Differential Geometry (MATH 5310) for 2.6. Invariance of the Interval. We use a similar language here to describe elements of $\mathbb{R}^4$ using the Lorentz metric tensor Here we are considering, in a sense, separation from the zero vector. For $\mathbf{x} \in \mathbb{R}^4$ we say:

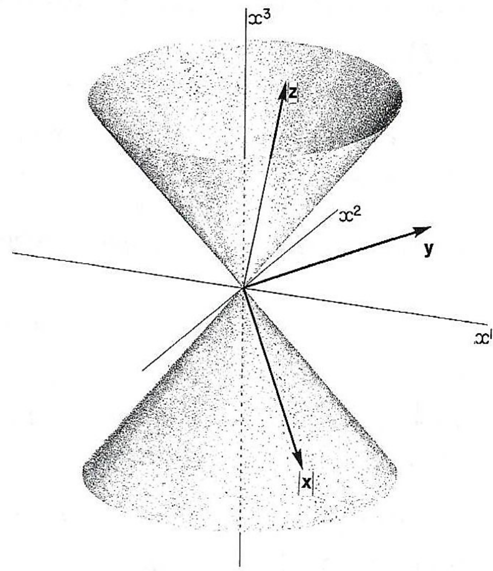
$$\mathbf{x} \text{ is } \textit{timelike} \text{ if } \mathbf{x} \cdot \mathbf{x} > 0,$$

$$\mathbf{x} \text{ is } \textit{spacelike} \text{ if } \mathbf{x} \cdot \mathbf{x} < 0,$$

$$\mathbf{x} \text{ is } \textit{lightlike} \text{ (or } \textit{null}\text{) if } \mathbf{x} \cdot \mathbf{x} = 0.$$

Example III.1.05. In order to draw light cones, we consider some examples similar to $\mathbb{R}^4$ under the Lorentz metric tensor. Let $\mathbb{H}^2 = \mathbb{R}^2$ with the metric tensor $(x^0, x^1) \cdot (y^0, y^1) = x^1 y^0 - x^1 y^1$. Let $\mathbb{H}^3 = \mathbb{R}^3$ with metric tensor $(x^0, x^1, x^2) \cdot$

$(y^0, y^1, y^2) = x^0y^0 - x^1y^1 - x^2y^2$. The null vectors in $\mathbb{H}^2$ are of the form (x, x) and $(x, -x)$. The null vectors in $\mathbb{H}^2$ are of the form (x^0, x^1, x^2) where $(x^0)^2 = (x^1)^2 + (x^2)^2$. This gives the *light cones*, timelike vectors, and spacelike vectors as follows (Figure 1.3 from page 69).



Note. We now define the familiar idea of a "traditional" norm on a vector space and consider some of its properties.

Definition IV.1.06. A *norm* on a vector space X is a function $\|\cdot\| : X \to \mathbb{R}$ such that for all $\mathbf{x}, \mathbf{y} \in X$ and $a \in \mathbb{R}$:

(N i) $\|\mathbf{x}\| = 0$ implies $\mathbf{x} = \mathbf{0}$.

(N ii) $\|\mathbf{x}a\| = |a|\|\mathbf{x}\|$.

(N iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (The Triangle Inequality).

A *partial norm* on a vector space X is a function $\|\cdot\| : X \to \mathbb{R}$ such that for all

$\mathbf{x} \in X$ and $a \in \mathbb{R}$,

(N′ i) $\|\mathbf{x}\| \geq 0$.

(N′ ii) $|\mathbf{x}a\| = |a|\|\mathbf{x}\|$.

Lemma IV.1.A. For $\|\cdot\|$ a norm on vector space X, we have $\|\mathbf{0}\| = 0$ and for all $\mathbf{x} \in X$ that $\|\mathbf{x}\| \geq 0$.

Note. Lemma IV.1.A shows that a norm satisfies Property (N′ i) in the definition of partial norm, so that a norm is also a partial norm. This is Exercise IV.1.7(a).

Definition. On an inner product space $(X, \mathbf{G})$ (remember that we take inner products to be nonnegative), define the *norm induced by* $\mathbf{G}$ as $|\mathbf{x}| = \|\mathbf{x}\|_{\mathbf{G}} = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. For metric vector space $(X, \mathbf{G}')$ (where $\mathbf{G}'$ may be indefinite), define the partial norm $\|\cdot\|_{\mathbf{G}'}$ as $\|\mathbf{x}\|_{\mathbf{G}'} = \sqrt{|\mathbf{G}'(\mathbf{x}, \mathbf{x})|}$.

Note. In Exercise IV.1.7(b), it is to be shown that $\|\cdot\|_{\mathbf{G}}$ is in fact a norm. In Exercise IV.1.7(c) it is to be shown that $\|\cdot\|_{\mathbf{G}'}$ is in fact a partial norm. If $\mathbf{G}'$ is an inner product (and hence nonnegative by our convention) then length and partial norm coincide: $|\cdot|_{\mathbf{G}'} = \|\cdot\|_{\mathbf{G}'}$.

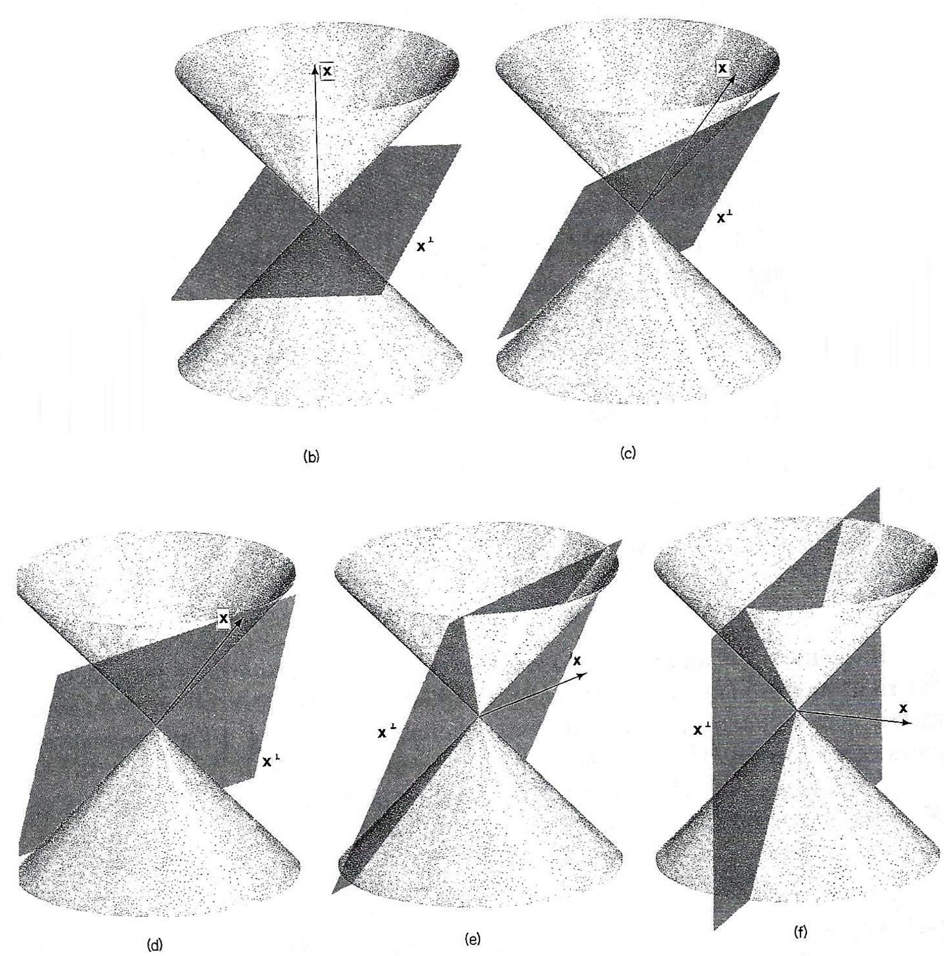
Definition. In a metric vector space $(X, \mathbf{G})$, we call $\|\mathbf{x}\|_{\mathbf{G}}$ (which we simply denote as $\|\mathbf{x}\|$) is the *size* of $\mathbf{x}$. A *unit vector* $\mathbf{x}$ satisfies $\|\mathbf{x}\| = 1$. A non-null vector is *normalized* as $\mathbf{x}/\|\mathbf{x}\|$.

Lemma IV.1.07. Schwarz's Inequality.
In any inner product space $(X, \mathbf{G})$ (with positive definite $\mathbf{G}$) we have for all $\mathbf{x}, \mathbf{y} \in X$ that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$ with equality for nonzero $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{y} = \mathbf{x}a$ for some $a \in \mathbb{R}$, and if $\mathbf{y} = \mathbf{x}a$ for some $a \geq 0$ then equality holds.

Note. We now consider some geometry based on orthogonality and perp spaces.

Definition IV.1.08. Two vectors $\mathbf{x}$ and $\mathbf{y}$ in a metric vector space are *orthogonal* if $\mathbf{x} \cdot \mathbf{y} = 0$. For any $\mathbf{x} \in X$, the set $\mathbf{x}^\perp$ of vectors orthogonal to it is the *perp space* (or *orthogonal complement*) of $\mathbf{x}$ (or sometimes "of the span of $\mathbf{x}$").

Note. It is clear that $\mathbf{x}^\perp$ actually is a subspace of X (we only need to show closure under linear combinations). If a metric vector space is in fact an inner product space then the perp space is as it is in $\mathbb{R}^n$. But in an indefinite metric space (such as Lorentz space with the Lorentz metric tensor), perp spaces can behave in new ways. Since $\mathbb{H}^3$ is 3 dimensional and $\mathbf{x}^\perp$ is a subspace of $\mathbb{H}^3$, then two linearly independent vectors in $\mathbf{x}^\perp$ determine $\mathbf{x}^\perp$. The following is Figure 1.5 from page 72.



- In Figure IV.1.5(b), we have $\mathbf{x} = (1,0,0)$. Then $(0,1,0)$ and $(0,0,1)$ are in $\mathbf{x}^\perp$ and are linearly independent. So $\mathbf{x}^\perp = \text{span}\{(0,1,0),(0,0,1)\}$ which is (with all vectors in "standard position") the x^1x^2-plane.

- In Figure IV.1.5(f), we have $\mathbf{x} = (0,1,0)$. Then $(1,0,0)$ and $(0,0,1)$ are in $\mathbf{x}^\perp$ and are linearly independent. So $\mathbf{x}^\perp = \text{span}\{)1,0,0),(0,0,1)\}$ which is the x^0x^2-plane.

- In Figure IV.1.5(d), we have $\mathbf{x} = (1,1,0)$. Then $(1,1,0)$ and $(0,0,1)$ are in $\mathbf{x}^\perp$ and are linearly independent. So $\mathbf{x}^\perp = \text{span}\{(1,1,0),(0,0,1)\} = \{(x^0,x^1,x^2) \in \mathbb{H}^3 \mid x^0 = x^1\}$. This is an unexpected result where $\mathbf{x} \in \mathbf{x}^\perp$.

- In Figure IV.1.5(c), say $\mathbf{x}$ is the future light cone of $(0,0,0)$, say $\mathbf{x} = (2,1,0)$. Then $(1,2,0)$ and $(0,0,1)$ are in $\mathbf{x}^\perp$ and are linearly independent. So $\mathbf{x}^\perp = \text{span}\{(1,2,0),(0,0,1)\} = \{(x^0,x^1,x^2) \in \mathbb{H}^3 \mid 2x^0 = x^1\}$.

- In Figure IV.1.5(e), we have $\mathbf{x}$ outside of both the future light cone and the past light cone of $(0,0,0)$, say $\mathbf{x} = (1,2,0)$. Then $(1,1/2,0)$ and $(0,0,1)$ are in $\mathbf{x}^\perp$ and are linearly independent. So $\mathbf{x}^\perp = \text{span}\{(1,1/2,0),(0,0,1)\} = \{(x^0,x^1,x^2) \in \mathbb{H}^3 \mid x^0 = 2x^1\}$.

We can think of Figures IV.1.5(b), (c), (d), (e), (f) as a "movie" as $\mathbf{x}$ rotates from vertical to horizontal. The respective perp spaces then rotate from horizontal to vertical (meeting up with $\mathbf{x}$ at the 45° mark).

Note. In a metric vector space X we can for $\mathbf{x} \in X$ define $\mathbf{x}^* \in X^*$ such that $\mathbf{x}^*(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for $\mathbf{y} \in X$. Notice that $\ker(\mathbf{x}^*) = \mathbf{x}^\perp$. Conversely, if $\mathbf{f} \in X^*$ then $\mathbf{f}(\mathbf{x}) = f_i x^i$ where $\mathbf{f} = f_i \mathbf{b}^i$ (by Note III.1.A), and so $\ker(\mathbf{f}) = \mathbf{x}^\perp$ where we take $\mathbf{x} = x^i \mathbf{b}_i = f_1\mathbf{b}_1 + f_2\mathbf{b}_2 + \cdots + f_n\mathbf{b}_n$. As the text says: "A metric tensor, then, gives us a geometrical way of changing from contravariant vector [such as $\mathbf{x}^*$ and $\mathbf{f}$] to covariant ones [such as $\mathbf{x}$ and $f_1\mathbf{b}_1 + f_2\mathbf{b}_2 + \cdots + f_n\mathbf{b}_n$] and vice versa." See page 73.

Theorem IV.1.09. For any non-degenerate bilinear form $\mathbf{F}$ on a vector space X, the map $\mathbf{F}_\downarrow : X \to X^*$ defined as $\mathbf{F}_\downarrow(\mathbf{x}) = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y}) = \mathbf{F}(\mathbf{x},\mathbf{y})$, is linear and an isomorphism.

Note IV.1.A. Since $\mathbf{F}_\downarrow : X \to X^*$ of Theorem IV.1.09 is one to one and onto, then $(\mathbf{F}_\downarrow)^{-1} = \mathbf{F}_\uparrow$ where $\mathbf{F}_\uparrow : X^* \to X$. In particular, in a metric vector space $(X, \mathbf{G})$, we have $\mathbf{G}_\downarrow : X \to X^*$ and $\mathbf{G}_\uparrow : X^* \to X$ as mappings induced by the metric tensor $\mathbf{G}$.

Note. The next result shows that a non-degenerate bilinear form on X induces a non-degenerate bilinear form on X^*.

Lemma IV.1.11. A non-degenerate bilinear form $\mathbf{F}$ on a vector space X induces a bilinear form $\mathbf{F}^*$ on X^* by

$$\mathbf{F}^*(\mathbf{f}, \mathbf{g}) = \mathbf{F}(\mathbf{F}_\uparrow(\mathbf{f}), \mathbf{F}_\uparrow(\mathbf{g}))$$

which is non-degenerate. In addition, if $\mathbf{F}$ is symmetric/anti-symmetric/positive definite/negative definite/indefinite then so is $\mathbf{F}^*$.

Note. As a special case of Lemma IV.1.11, we can take $\mathbf{F}$ as a metric tensor (i.e., a symmetric non-degenerate bilinear form) or an inner product (i.e., a positive or negative definite metric tensor). This gives the following.

Corollary IV.1.12. A metric tensor $\mathbf{G}$ on X induces a metric tensor $\mathbf{G}^*$ on X^*. An inner product $\mathbf{G}$ on X induces an inner product $\mathbf{G}^*$ on X^*.

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