IV.2. Maps

Note. In this section we consider orthogonal projections, isometries, and adjoints.

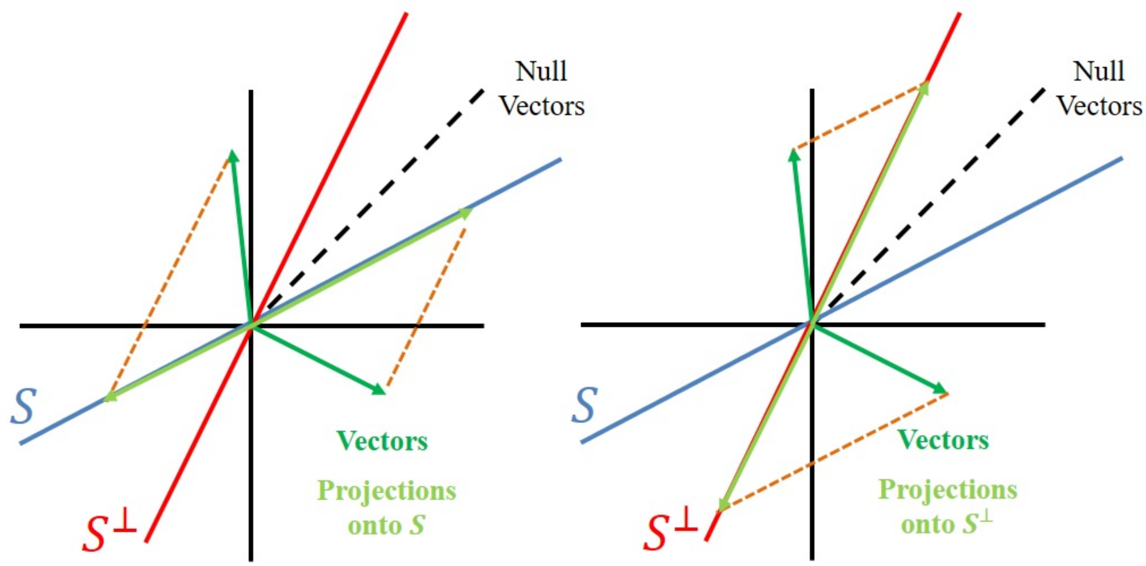
Theorem IV.2.01. Let S be a non-degenerate subspace of a metric vector space X. Then there is a unique linear operator $\mathbf{P} : X \to S$ such that $(\mathbf{x} - \mathbf{Rx}) \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in S$.

Definition. The linear operator $\mathbf{R} : X \to X$ of Theorem IV.2.01 is the *orthogonal projection onto* S. For $\mathbf{x} \in X$, $\mathbf{Rx}$ is the *component of* $\mathbf{x}$ *in* S and $\mathbf{x} - \mathbf{Px}$ is the *component of* $\mathbf{x}$ *orthogonal to* S.

Corollary IV.2.02. The projection operator $\mathbf{P}$ onto S is idempotent. That is, $\mathbf{P}(\mathbf{Px}) = \mathbf{Px}$ for all $\mathbf{x} \in X$.

Note. In Linear Algebra (MATH 2010) we consider projections of a vector $\mathbf{b}$ onto a subspace W of $\mathbb{R}^n$, denoted $\text{proj}_W(\mathbf{b})$. See my online notes for Linear Algebra on "6.1. Projections" where this is addressed computationally and geometrically. This is addressed in the setting of linear mappings (that is, matrices) in "6.4 Projection Mappings". If $W = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k)$ is a k-dimensional subspace of $\mathbb{R}^n$ and A is the matrix with the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ as its columns, then the matrix $P = A(A^tA)^{-1}A^t$ is the matrix that produces projections onto W. That is, $\text{proj}_X(\mathbf{b}) = P\mathbf{b}$. This is the same situation we have here when we consider $(X, G) = (\mathbb{R}, \langle \cdot, \cdot \rangle)$.

Note. In $\mathbb{R}^n$ orthogonality is as expected, and so are projections. However, in $\mathbb{H}^2$ orthogonality is somewhat strange (as illustrated in the larger $\mathbb{H}^3$ in Figure IV.1.5). But we still have $\mathbf{x} - \mathbf{Px}$ as orthogonal to subspaces. A subspace of $\mathbf{H}^2$ is a line through the origin (when all vectors are placed in standard position with their tails at the origin). For $X = \{(x^0, 0) \in \mathbb{H}^2 \mid x^0 \in \mathbb{R}\}$ a subspace of $\mathbb{H}^2$ and projection onto it are as they are in $\mathbb{R}^2$ since the space of all vectors perpendicular to S is $S^\perp = \{(0, x^1) \in \mathbb{H}^2 \mid x^1 \in \mathbb{R}\}$. Similarly, projection onto $S^\perp$ are as they are in $\mathbb{R}^2$. However, on $S = \{(x^0, x^1) \in \mathbb{H}^2 \mid x^0 = 2x^1\}$ we have the space of all vectors perpendicular to S as $S^\perp = \{(x^0, x^1) \in \mathbb{H}^2 \mid x^1 = 2x^0\}$ and so $\mathbf{x} - \mathbf{Px} \in S^\perp$ when $\mathbf{P}$ projects onto S. We then get geometrically:



Note. We have referred to a perp space already, but we have not yet formally defined it. We do so now.

Definition IV.2.03. The kernel of the orthogonal projection onto a non-degenerate subspace S of X is called the *orthogonal complement* of S in X, denoted $S^\perp$.

Note. The next result is a common geometric result when speaking of subspaces and orthogonal complements.

Lemma IV.2.04. For any non-degenerate subspace S of X, each $x \in X$ can be uniquely expressed as $\mathbf{x} = \mathbf{s} + \mathbf{t}$ where $\mathbf{x} \in S$ and $\mathbf{f} \in S^\perp$.

Definition. For S a non-degenerate subspace of X and $S^\perp$ the orthogonal complement of S, we say that X is the *direct sum* of S and $S^\perp$, denoted $X = S \oplus S^\perp$.

Note. For $X = S \oplus S^\perp$ we see be Lemma IV.2.04 that each $\mathbf{x} \in X$ can be written as $\mathbf{x} = \mathbf{s} + \mathbf{t}$ for unique $\mathbf{s} \in S$ and $\mathbf{t} \in S^\perp$. The proof of the following is to be given in Exercise IV.2.1(c).

Corollary IV.2.05. If $\dim(S) = k$ and $\dim(X) = n$, then $\dim(S^\perp) = n - k$.

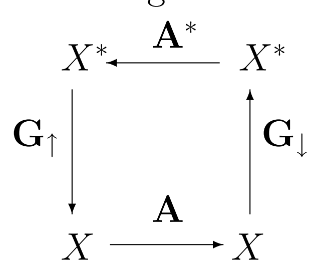
Corollary IV.2.06. If G is non-degenerate on S, it is non-degenerate on $S^\perp$.

Note. Since $\mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ is a metric tensor on vector space X, then $\mathbf{G}$ (and the dot product) determine "distances," so if a linear map preserves dot products (and values of $\mathbf{G}$) then it preserves 'distances." We therefore have the following definition.

Definition IV.2.07. A linear map $\mathbf{A} : X \to Y$ between metric vector spaces is an *isometry* if it is onto (surjective) and $\mathbf{G}(\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x}') = (\mathbf{A}\mathbf{x})\cdot(\mathbf{A}\mathbf{x}') = \mathbf{G}(\mathbf{x}, \mathbf{x}') = \mathbf{x}\cdot\mathbf{x}'$ for all $\mathbf{x}, \mathbf{x}' \in X$.

Note. An isometry is also called *orthogonal* (or *unitary*). This terminology (especially "orthogonal") is due to the fact that in $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ such linear transformations are represented by matrices with orthonormal rows and orthonormal columns. See "Theorem 6.5. Characterizing Properties of an Orthogonal Matrix" in my online Linear Algebra notes for 6.3. Orthogonal Matrices. This result also shows the relationship $A^{-1} = A^t$, which we establish in our setting in Lemma IV.2.09.

Note. Let $\mathbf{A}$ be a linear operator on a metric vector space $(X, \mathbf{G})$. Denote $\mathbf{A}^T = \mathbf{G}_\uparrow \mathbf{A}^* \mathbf{G}_\downarrow$. Notice that we have the diagram:



So $\mathbf{A}^T : X \to X$, as it should. So for any $\mathbf{x} \in X$ we have $\mathbf{A}^T(\mathbf{x}) = \mathbf{G}_\uparrow \mathbf{A}^* \mathbf{G}_\downarrow(\mathbf{x})$ or $(\mathbf{A}^T(\mathbf{x}) = \mathbf{G}_\uparrow(\mathbf{A}^*(\mathbf{G}_\downarrow(\mathbf{x})))$. Applying $\mathbf{G}_\downarrow$ to both sides of this equation (and observing that $\mathbf{G}_\downarrow = (\mathbf{G}_\uparrow)^{-1}$ (see the Note after Theorem IV.1.09) we have

$$\mathbf{G}_\downarrow(\mathbf{A}^T(\mathbf{x})) = \mathbf{G}_\downarrow(\mathbf{G}_\uparrow(\mathbf{A}^*(\mathbf{G}_\downarrow(\mathbf{x})))) = \mathbf{A}^*(\mathbf{G}_\downarrow(\mathbf{x})).$$

Now $\mathbf{A}^*\mathbf{G}_\downarrow(\mathbf{x})) \in X^*$, so we can apply it to any $\mathbf{y} \in X$, and we have

$$\mathbf{G}_\downarrow(\mathbf{A}^T(\mathbf{x}))(\mathbf{y} = \mathbf{A}^*(\mathbf{G}_\downarrow(\mathbf{x}))(\mathbf{y}). \qquad (*)$$

By the definition of dual map (form Section III.1), $\mathbf{A}^*(\mathbf{f}) = \mathbf{f} \circ \mathbf{A}$ for $\mathbf{f} \in X^*$, so with $\mathbf{G}_\downarrow(\mathbf{x}) \in X^*$ we see that $\mathbf{A}^*(\mathbf{G}_\downarrow(\mathbf{x})) = \mathbf{G}_\downarrow(\mathbf{x})(\mathbf{A})$. Then from $(*)$ we have

$$\mathbf{G}_\downarrow(\mathbf{A}^T(\mathbf{x}))(\mathbf{y}) = \mathbf{G}_\downarrow(\mathbf{x})(\mathbf{A}\mathbf{y}). \qquad (**)$$

Now by Theorem IV.1.09, $\mathbf{G}_\downarrow(\mathbf{x}) = \mathbf{x}^*$ where $\mathbf{x}^*(\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$, so $\mathbf{G}_\downarrow(\mathbf{A}^T(\mathbf{x}))(\mathbf{y} = \mathbf{A}^T(\mathbf{x}) \cdot \mathbf{y}$ and $\mathbf{G}_\downarrow(\mathbf{x})(\mathbf{A}\mathbf{y}) = \mathbf{x} \cdot \mathbf{A}\mathbf{y}$, So $(**)$ implies that $\mathbf{A}^T\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{A}\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in X$. We use this property as the formal definition of $\mathbf{A}^T$.

Definition IV.2.08. The *adjoint* $\mathbf{A}^T$ of a linear operator $\mathbf{A}$ on a metric vector space $(X, \mathbf{G})$ is defined by the equation $\mathbf{A}^T\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{A}\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in X$. (We see from the previous not that $\mathbf{A}^T$ exists, is unique, and is given by $\mathbf{A}^T = \mathbf{G}_\uparrow\mathbf{A}^*\mathbf{G}_\downarrow$.) An operator $\mathbf{A}$ on X is *self adjoint* if $\mathbf{A}^T = \mathbf{A}$.

Lemma IV.2.A. Properties of Adjoint.

For $\mathbf{A}$ and $\mathbf{B}$ linear operators on a metric vector space (X, G) we have:

(a) $\mathbf{I}^T = \mathbf{I}$ where $\mathbf{I}$ is the identity operator.

(b) $(\mathbf{A}^T)^T = \mathbf{A}$.

(c) $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$.

Lemma IV.2.09. An operator $\mathbf{A}$ on a metric vector space $(X, \mathbf{G})$ is orthogonal if and only if $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.

Note. Dodson and Poston states the following without proof. Your humble instructor thinks this could use some justification (we don't necessarily have commutivity and can't immediately conclude that $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A}$.

Corollary IV.2.10. A linear operator $\mathbf{A}$ on a metric vector space $(X, \mathbf{G})$ is orthogonal if and only if $\mathbf{A}^T$ is orthogonal.

Note. Orthogonal projections are examples of self adjoint linear operators, as we now show.

Lemma IV.2.11. Orthogonal projection $\mathbf{P}$ onto a non-degenerate subspace S of a metric vector space X is a self adjoint operator.

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