Chapter VII. Differentiation and Manifolds

Note. In this chapter, X and X' are affine spaces of dimension n and m respectively with respective difference functions $\mathbf{d}$ and $\mathbf{d}'$ and respective vector spaces T and T'.

VII.1. Differentiation (Partial)

Note. For smooth $f : \mathbb{R} \to \mathbb{R}$, the derivative function f' determines a tangent line to the (graph) of function f at a given point x. In higher dimensions, even for a surface with a 2-dimensional plane as a tangent space, differentiation is less clearly related to tangents. For f a map between affine spaces X and X' we think of a derivative as a first order approximation or a linearization, as follows.

Definition VII.1.01. If $f : X \to X'$ is a map (not necessarily affine) between affine spaces, a *derivative* of f at $x \in X$ is a linear map of the tangent space at x, $\mathbf{D}_n f : T_x X \to T_{f(x)} X'$, such that for any neighborhood N in $L(T_x X; T_{f(x)} X)$ of the zero linear map, there is a neighborhood N' of $\mathbf{0} \in T_x X$ such that if $\mathbf{t} \in N'$ then

$$\mathbf{d}'(f(x + \mathbf{t}), f(x)) = \mathbf{d}'_{f(x)}(\mathbf{D}_x f(\mathbf{t}) + \mathbf{A}(\mathbf{t}))$$

for some $\mathbf{A} \in N$. If f has a derivative at x then f is *differentiable at* x. If f is differentiable at all x for which f is defined then f is said to be *differentiable.*

Note. Definition VII.1.01 is just an ε/δ type definition based on the topologies on X and X'. The "for all $\varepsilon > 0$" idea is replaced with "for any neighborhood $N \ldots$ of the zero linear map" and the "there exists $\delta > 0$" idea is replaced with "there is a neighborhood N' of $\mathbf{0}$." Think of $x + \mathbf{t}$ and x as within δ of each other and $\mathbf{A}(\mathbf{t})$ within ε of the zero map.

Note. In Exercise VII.1.3(a) it is to be shown that if f is differentiable at x then the derivative of f is unique. In Exercise VII.1.3(b) it is to be shown that the derivative can be written as a limit:

$$\mathbf{D}_x f(\mathbf{t}) = \lim_{h \to 0} \mathbf{D}'^{-1}_{f(x)} \frac{\mathbf{d}'(f(x), f(x + h\mathbf{t}))}{h}.$$

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