VII.2. Manifolds

Note. In this section, we define a manifold modeled on an affine space, chart/atlas, mappings between manifolds, and tangent spaces to a manifold.

Note. We desire a "local resemblance" to an affine plane for a manifold so that, as seen in the previous section, we can then set up differentiation of functions defined on the manifold.

Definition VII.2.01. A C^k-*manifold* modeled on an affine space X is a Hausdorff topological space M together with a collection of open sets $\{U_a \mid a \in A\}$ in M and corresponding maps $\varphi_a : U_a \to X$ such that

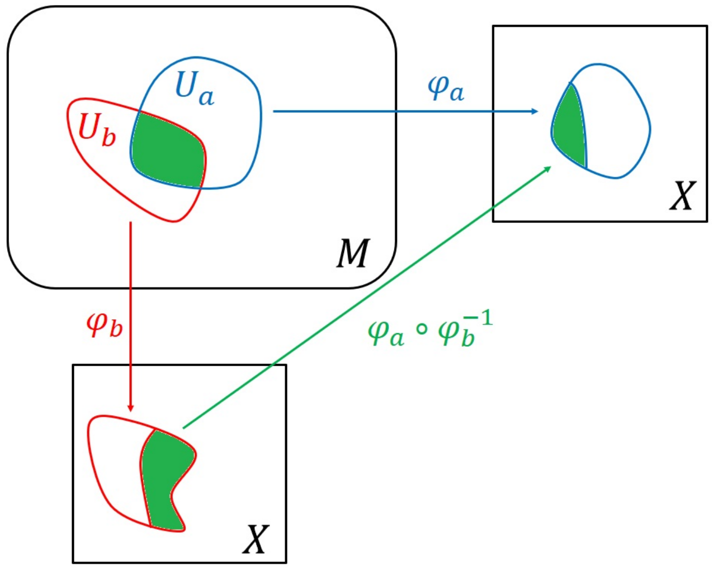
(M i) $\cup_{a \in A} U_a = M$.

(M ii) Each φ_a defines a homeomorphism mapping U_a to $\varphi_a(U_a) \subseteq X$.

(M iii) If $U_a \cap U_b \neq \varnothing$ then the composites $\varphi_a \circ \varphi_b^{-1} : X \to X$ and $\varphi_b \circ \varphi_a^{-1} : X \to X$ on the sets $\varphi_b(U_a) \subseteq X$ and $\varphi_a(U_b) \subseteq X$ (respectively) on which they are defined are C^k.

The pairs (U_a, φ_a) are called *charts* on M and the set $\{(U_1, \varphi_a) \mid a \in A\}$ of all such charts is an *atlas*. The *dimension* of M is the dimension of X, $\dim(M) = \dim(X)$. If $\dim(M) = n$ then M is called an *n-manifold*.

Note. Notice that $\varphi_a \circ \varphi_b^{-1} : X \to X$ and $\varphi_b \circ \varphi_a^{-1} : X \to X$ so the continuity and differentiability of these maps is defined in the previous section.



Note. Dodson and Poston make insightful comments about the definition of a manifold. "The axioms (M i)–(M iii) are natural enough; (M i) just says that no point in M is 'uncharted,' (M ii) that the charts are topologically uncomplicated, relative to the topology on M, and (M iii) that they are differentially nice (C^k) relative to each other." See page 161.

Definition. A new chart (U, φ) (that is, a chart not already in the atlas of M) is *admissible* if for all $a \in A$ we have $\varphi \circ \varphi_a^{-1} : X \to X$ and $\varphi_a \circ \varphi^{-1} : X \to X$ are C^k whenever they are defined (that is, on the appropriate domains).

Note. It is common to require the atlas of a manifold to be "maximal." That is, it includes all admissible charts. See my online notes based on Wald's *General Relativity* on "2.1. Manifolds" (see the convention introduced after the definition of manifold). Dodson and Poston say as much themselves when they state (see page 161): "M is not changed in any significant way if we enlarge the family $(U_a, \varphi_a) \mid a \in A\}$ by adding admissible charts, and we shall feel free to do so."

Note. In Dodson and Poston, the term "manifold" is used, unless stated otherwise, to mean a C^∞-manifold (which they also call a "smooth manifold").

Note. We can use charts on manifolds to define differentiability of functions between manifolds. The following definition refers to "some charts", but in Exercise VII.2.2 it is to be shown that the definition is independent of the charts used (and so the condition holds for all admissible charts). Recal from Definition VI.1.11 that a homeomorphism f between topological spaces is a continuous bijection with a continuous inverse. Since manifolds are, by definition, topological spaces then the concept of a homeomorphism between manifolds is defined.

Definition VII.2.02. A map $f : M \to N$ between smooth manifolds is *differentiable* (respectively, C^k) at $x \in M$ if for some charts (U, φ) on M and (V, ψ) on N, with $x \in U$ and $f(x) \in V$, the map $\psi \circ f \circ \varphi^{-1} : X \to X$ (notice $\varphi^{-1} : X \to M$, $f : M \to N$, and $\psi : N \to X$) is differentiable (respectively C^k) at $\varphi(x) \in X$. A homeomorphism $f : M \to N$ between C^k manifolds is a C^k *diffeomorphism* if both f and f^{-1} are C^k. If there is a diffeomorphism between two manifolds they are *diffeomorphic*.

Note. In the geometric discussion of tangent spaces to a manifold, Dodson and Poston mention the Whitney Embedding Theorem which state that an m-manifold (along with its "differential structure") can be embedded in $\mathbb{R}^n$ where $n \leq 2m+1$. For example, a Klein bottle (where $m = 1$) can be embedded in $\mathbb{R}^4$. With such an embedding for manifold M, with $x \in M$ we can define the tangent space at x geometrically as the affine subspace $\tilde{T}_x M \subseteq \mathbb{R}^n$ (using the geometry of $\mathbb{R}^n$). More precisely, to get a tangent vector space as in Definition II.1.02, we consider $T_x(\tilde{T}_x M)$. We could denote this as $T_x(M)$; see page 165. However, this argument depends on the embedding (a term we have not defined). We want a definition of tangent space depending only on the properties of the manifold itself (namely, the open sets and the charts). This is done with Exercise VII.2.3.

Definition. Let M be a manifold modeled on affine space X with vector space T. For charts (U, φ) and (U', φ') on M, $u \in U \cap U'$, and $\mathbf{t}, \mathbf{t}' \in T$, define the *relation* $\sim$ by

$$(U, \varphi, \mathbf{t}) \sim (U', \varphi', \mathbf{t}') \text{ if and only if } \hat{\mathbf{D}}_{\varphi(u)}(\varphi' \circ \varphi^{-1})\mathbf{t} = \mathbf{t}'.$$

Note. By Exercise VII.2.3(a), relation $\sim$ is an equivalence relation. By Exercise VII.2.3(b), if $(U, \varphi, \mathbf{t}) \sim (U', \varphi', \mathbf{t}')$ and $(U, \varphi, \mathbf{s}) \sim (U', \varphi', \mathbf{s}')$ then $(U, \varphi, \mathbf{t} + \mathbf{s}) \sim (U', \varphi', \mathbf{t}' + \mathbf{s}')$ and for all $a \in \mathbb{R}$ $(U, \varphi, \mathbf{t}a) \sim (U', \varphi', \mathbf{t}'a)$. So we can define addition and scalar multiplication on equivalence classes of

$$T_u M = \{(U, \varphi, \mathbf{t}) \mid (U, \varphi) \text{ is a chart on } M \text{ with } u \in U \text{ and } \mathbf{t} \in T\},$$

making $T_u M$ a vector space.

Definition. Let M be a smooth manifold modeled on affine space X with vector space T. Define addition and scalar multiplication on the equivalence classes of T_uM as above. Then T_uM is the *tangent space to* M *at* u and the equivalence classes of T_uM are *tangent vectors* to M at u.

Note. Now if M and N are smooth manifolds and $f : M \to N$ then we can interpret the derivative of f at x as the linear map $\mathbf{D}_x f : T_xM \to T_{f(x)}N$ as explained in Exercise VII.2.6 as follows.

Note. In Exercise VII.2.6(a) it is to be shown that if $f : M \to N$ is differentiable at $u \in U \cap U'$ and (V, ψ) is a chart on N with $f(u) \in V$, then $(U, \varphi, \mathbf{t}) \sim (U', \varphi', \mathbf{t}')$ implies $\mathbf{D}_{\varphi(u)}(\psi \circ f \circ \varphi^{-1})\mathbf{t} = \mathbf{D}_{\varphi'(u)}(\psi \circ f \circ \varphi'^{-1})\mathbf{t}'$ which in turn implies that $(V, \psi, \mathbf{D}_{\varphi(u)}(\psi \circ f \circ \varphi^{-1})\mathbf{t}) \sim (V, \psi, \mathbf{D}_{\varphi'(u)}(\psi \circ f \circ \varphi'^{-1})\mathbf{t}')$ so that f induces a well defined map $\mathbf{D}_u f : T_uM \to T_{f(u)}N$ which takes the equivalence class of $(U, \varphi, \mathbf{t})$ to the equivalence class of $(V, \psi, \mathbf{D}_{\varphi(u)}(\psi \circ f \circ \varphi^{-1})\mathbf{t})$, and $\mathbf{D}_u f$ is linear. In Exercise VII.2.6(b), it is shown that we can differentiate $\varphi : U \to X$ at $u \in U$ by mapping each $\mathbf{t} \in T_xM$ to its representative in $T_{\varphi(x)}X$; this map is denoted $\mathbf{D}_x\varphi$.

Note. Dodson and Poston conclude this section by observing that for each $x \in M$, T_xM is isomorphic to T. However, unlike with the case that there was a natural isomorphism $\mathbf{d}_x$ in the affine space setting where $T_xX \cong T$, there is no one "natural" isomorphism between T_xM and T; in fact, each chart (U, φ) with $x \in U$ gives such an isomorphism; namely $\mathbf{d}_{\varphi(x)} \circ \mathbf{D}_x\varphi : T_xM \to T$. See page 166.

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