Chapter 1. Complex Numbers
Section 1.2. Basic Algebraic Properties—Proofs of Theorems

Theorem 1.2.1

Theorem 1.2.1. For any $z_1, z_2, z_3 \in \mathbb{C}$ we have the following.

1. Commutativity of addition and multiplication:
   
   \[ z_1 + z_2 = z_2 + z_1 \text{ and } z_1z_2 = z_2z_1. \]

2. Associativity of addition and multiplication:
   
   \[ (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \text{ and } (z_1z_2)z_3 = z_1(z_2z_3). \]

3. Distribution of multiplication over addition:
   
   \[ z_1(z_2 + z_3) = z_1z_2 + z_1z_3. \]

4. There is an additive identity $0 = 0 + i0$ such that $0 + z = z$ for all $z \in \mathbb{C}$. There is a multiplicative identity $1 = 1 + i0$ such that $z_1 = z$ for all $z \in \mathbb{C}$. Also, $z0 = 0$ for all $z \in \mathbb{C}$.

5. For each $z \in \mathbb{C}$ there is $z' \in \mathbb{C}$ such that $z' + z = 0$. $z'$ is the additive inverse of $z$ (denoted $-z$). If $z \neq 0$, then there is $z'' \in \mathbb{C}$ such that $z''z = 1$. $z''$ is the multiplicative inverse of $z$ (denoted $z^{-1}$).

Proof. We have abandoned the ordered pair notation, so let $z_k = x_k + iy_k$ for $k = 1, 2, 3$ and let $z = x + iy$, where $x_k, y_k, x, y \in \mathbb{R}$.

1. (Commutativity) For addition, we have
   
   \[
   z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) \\
   = (x_1 + x_2) + i(y_1 + y_2) \text{ by the definition of } + \text{ in } \mathbb{C} \\
   = (x_2 + x_1) + i(y_2 + y_1) \text{ since } + \text{ is commutative in } \mathbb{R} \\
   = (x_2 + iy_2) + (x_1 + iy_1) \text{ by the definition of } + \text{ in } \mathbb{C} \\
   = z_2 + z_1.
   \]

2. (Associativity) For addition we have
   
   \[
   (z_1 + z_2) + z_3 = ((x_1 + iy_1) + (x_2 + iy_2)) + (x_3 + iy_3) \\
   = ((x_1 + x_2) + i(y_1 + y_2)) + (x_3 + iy_3) \text{ by the definition of } + \text{ in } \mathbb{C}
   \]

\[
= \]
Theorem 1.2.1 (continued 3)

\[(z_1 + z_2) + z_3 = ((x_1 + x_2) + x_3) + i((y_1 + y_2) + y_3)\] by the definition of $+$ in $\mathbb{C}$
\[= (x_1 + (x_2 + x_3)) + i(y_1 + (y_2 + y_3))\] since $+$ is associative in $\mathbb{R}$
\[= (x_1 + iy_1) + ((x_2 + x_3) + i(y_2 + y_3))\] by the definition of $+$ in $\mathbb{C}$
\[= z_1 + (z_2 + z_3)\]

For multiplication we have

\[(z_1 z_2) z_3 = (x_1 x_2 - y_1 y_2)(x_3 + iy_3)\]
\[= ((x_1 x_2 - y_1 y_2)(x_3 + iy_3))\] by the definition of $\cdot$ in $\mathbb{C}$
\[= z_1(z_2 z_3)\]

Theorem 1.2.1 (continued 5)

3. (Distribution) For distribution we have

\[z_1(z_2 + z_3) = (z_1 + iy_1)((x_2 + iy_2) + (x_3 + iy_3))\]
\[= (x_1 + iy_1)((x_2 + x_3) + i(y_2 + y_3))\] by the definition of $+$ in $\mathbb{C}$
\[= (x_1(x_2 + x_3) - y_1(y_2 + y_3)) + i(y_1(x_2 + x_3) + x_1(y_2 + y_3))\] by the definition of $\cdot$ in $\mathbb{C}$
\[= (x_1 x_2 + x_1 x_3 - y_1 y_2 - y_1 y_3) + i(y_1 x_2 + x_1 x_3 + x_1 y_2 + x_1 y_3)\] by distribution in $\mathbb{R}$
\[= (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_3) + i(x_1 x_3 - y_1 y_3)\] by the definition of $+$ in $\mathbb{C}$
\[= (x_1 + iy_1)(x_2 + iy_2) + (x_1 + iy_1)(x_3 + iy_3)\] by the definition of $\cdot$ in $\mathbb{C}$
\[= z_1 z_2 + z_1 z_3\]

Theorem 1.2.1 (continued 6)

4. (Identities) We easily have

\[0 + z = (0 + 0) + (x + iy)\]
\[= (0 + x) + i(0 + y)\] by the definition of $+$ in $\mathbb{C}$
\[= x + iy\] since 0 is the additive identity in $\mathbb{R}$
\[= z\]

and

\[1z = (1 + 0) + (x + iy)\]
\[= ((1)(x) - (0)(y)) + i((0)(x) + (1)(y))\] by the definition of $\cdot$ in $\mathbb{C}$
\[= x + iy\] since 1 is the multiplicative identity in $\mathbb{R}$
\[= z.\]
Also, 
\[ \begin{align*}
  z0 &= (x + iy)(0 + i0) \\
 &= ((x)(0) - (y)(0)) + i((y)(0) + (x)(0)) \\
 &= 0 + i0 \
& \text{by the definition of } \cdot \text{ in } \mathbb{C} \\
& = 0 + i0 \text{ since } r0 = 0 \text{ for all } r \in \mathbb{R} \\
& = 0.
\end{align*} \]

5. **(Inverses)** For \( z = x + iy \), we take \( z' = (-x) + i(-y) \) and then 
\[ \begin{align*}
  z + z' &= (x + iy) + ((-x) + i(-y)) \\
& = (x + (-x)) + i(y + (-y)) \text{ by the definition of } + \text{ in } \mathbb{C} \\
& = 0 + i0 \text{ since } -x \text{ and } -y \text{ are the } + \text{ inverses of } x \text{ and } y, \\
& \text{respectively, in } \mathbb{R} \\
& = 0.
\end{align*} \]

**Corollary 1.2.2.** For \( z_1, z_2, z_3 \in \mathbb{C} \) if \( z_1z_2 = 0 \) then either \( z_1 = 0 \) or \( z_2 = 0 \). That is, \( \mathbb{C} \) has no “zero divisors.”

**Proof.** Suppose one of \( z_1 \) or \( z_2 \) is nonzero. WLOG, say \( z_1 \neq 0 \). Then, by Theorem 1.2.1(5), there is a multiplicative inverse \( z_1^{-1} \in \mathbb{C} \) such that \( z_1z_1^{-1} = 1 \). So 
\[ \begin{align*}
  z_2 &= z_21 \text{ by Theorem 1.2.1(4) } (\cdot \text{ identity}) \\
& = z_2(z_1z_1^{-1}) \\
& = (z_1z_1^{-1})z_2 = (z_1^{-1}z_1)z_2 \text{ by Theorem 1.2.1(1) } (\text{commutivity of } \cdot) \\
& = z_1^{-1}(z_1z_2) \text{ by Theorem 1.2.1(2) } (\text{associativity of } \cdot) \\
& = z_1^{-1}(0) \text{ by hypothesis} \\
& = 0 \text{ by Theorem 1.2.1(4)}. \\
\end{align*} \]

So if one of \( z_1, z_2 \) is nonzero, then the other is 0 and the result follows. \( \square \)