Theorem 1.7.1

Theorem 1.7.1. For $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2} \in \mathbb{C}$ we have

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \text{ and } \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

Proof. First, notice that

$$e^{i\theta_1} e^{i\theta_2} = (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$
$$= (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)$$
$$= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$$
$$= e^{i(\theta_1 + \theta_2)}.$$

So

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1 + \theta_2)} = (r_1) e^{i(\theta_1 + \theta_2)}.$$

Corollary 1.7.2

Corollary 1.7.2. If $z = r e^{i\theta} \in \mathbb{C}$, then for $n \in \mathbb{Z}$ we have $z^n = r^n e^{i n \theta}$.

Proof. For $n > 0$, we use mathematical induction. First, for the base case $n = 1$, the result is trivial. Now suppose the result holds for $n = m$; that is, suppose $z^m = r^m e^{i m \theta}$ (this is the "induction hypothesis"). To complete the induction argument, we must show the result holds for $n = m + 1$. So consider

$$z^{m+1} = z^m z = (r^m e^{i m \theta}) z \text{ by the induction hypothesis}$$
$$= r^m e^{i m \theta} r e^{i \theta} = r^m r e^{i m \theta} e^{i \theta}$$
$$= r^{m+1} e^{i (m+1) \theta} \text{ by Theorem 1.7.1}.$$

So the result holds for all $n > 0$.

For $n = 0$, we have $z^0 = 1$ (by "convention," provided $z \neq 0$) and $1 = r^0 e^{i0}$, so the result holds for $n = 0$.
Corollary 1.7.2. If \( z = re^{i\theta} \in \mathbb{C} \), then for \( n \in \mathbb{Z} \) we have \( z^n = r^n e^{in\theta} \).

**Proof (continued).** For \( n < 0 \), let \( m = -n \) (so \( m > 0 \)) and

\[
\begin{align*}
z^n &= z^{-m} = (z^{-1})^m = (r^{-1}e^{-i\theta})^m \quad \text{by the note above} \\
&= r^{-m}(e^{-i\theta})^m = \left(\frac{1}{r}\right)^m \frac{1}{(e^{i\theta})^m} \\
&= \frac{1}{r^m e^{im\theta}} \quad \text{by the first part of the proof, since } m > 0 \\
&= r^{-m}e^{i(-m)\theta} = r^n e^{in\theta}.
\end{align*}
\]

So the result holds for all \( n < 0 \) and hence holds for all \( n \in \mathbb{Z} \). \( \square \)

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**Corollary 1.7.3.** For all \( n \in \mathbb{Z} \), we have

\[ (\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta). \]

**Proof.** Since \( e^{i\theta} = \cos \theta + i \sin \theta \), then

\[
\begin{align*}
(e^{i\theta})^n &= (\cos \theta + i \sin \theta)^n \\
&= e^{in\theta} \quad \text{by Corollary 1.7.2} \\
&= \cos(n\theta) + i \sin(n\theta).
\end{align*}
\]

\( \square \)