Chapter 1. Complex Numbers
Section 1.8. Arguments of Products and Quotients—Proofs of Theorems
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Lemma 1.8.1

**Lemma 1.8.1.** For any $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq 0, z_2 \neq 0$, we have $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

**Proof.** Let $\theta_1 + \theta_2 \in \arg(z_1) + \arg(z_2)$ (so that $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$). Then $z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2)e^{i(\theta_1 + \theta_2)}$ and so $\theta_1 + \theta_2 \in \arg(z_1 z_2)$. So $\arg(z_1) + \arg(z_2) \subset \arg(z_1 z_2)$. 


Lemma 1.8.1

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**Proof.** Let \( \theta_1 + \theta_2 \in \arg(z_1) + \arg(z_2) \) (so that \( z_1 = r_1e^{i\theta_1} \) and \( z_2 = r_2e^{i\theta_2} \)). Then \( z_1z_2 = (r_1e^{i\theta_1})(r_2e^{i\theta_2}) = (r_1r_2)e^{i(\theta_1+\theta_2)} \) and so \( \theta_1 + \theta_2 \in \arg(z_1z_2) \). So \( \arg(z_1) + \arg(z_2) \subset \arg(z_1z_2) \).

Now let \( \theta \in \arg(z_1z_2) \). Then \( z_1z_2 = r_1r_2e^{i\theta} \) where \( r_1 = |z_1| \) and \( r_2 = |z_2| \). Let \( \theta_1 \in \arg(z_1) \). Then \( z_1 = r_1e^{i\theta_1} \). Let \( \theta_2 = \theta - \theta_1 \). Then

\[
    r_2e^{i\theta_2} = r_2e^{i(\theta-\theta_1)} = r_2e^{i\theta}e^{-i\theta_1} = \frac{r_1r_2e^{i\theta}}{r_1e^{i\theta_1}} = \frac{z_1z_2}{z_1} = z_2
\]

and so \( \theta_2 \in \arg(z_2) \).
Lemma 1.8.1

Lemma 1.8.1. For any \( z_1, z_2 \in \mathbb{C} \) with \( z_1 \neq 0, z_2 \neq 0 \), we have \( \arg(z_1z_2) = \arg(z_1) + \arg(z_2) \).

Proof. Let \( \theta_1 + \theta_2 \in \arg(z_1) + \arg(z_2) \) (so that \( z_1 = r_1 e^{i\theta_1} \) and \( z_2 = r_2 e^{i\theta_2} \)). Then \( z_1z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \) and so \( \theta_1 + \theta_2 \in \arg(z_1z_2) \). So \( \arg(z_1) + \arg(z_2) \subset \arg(z_1z_2) \).

Now let \( \theta \in \arg(z_1z_2) \). Then \( z_1z_2 = r_1 r_2 e^{i\theta} \) where \( r_1 = |z_1| \) and \( r_2 = |z_2| \). Let \( \theta_1 \in \arg(z_1) \). Then \( z_1 = r_1 e^{i\theta_1} \). Let \( \theta_2 = \theta - \theta_1 \). Then

\[
    r_2 e^{i\theta_2} = r_2 e^{i(\theta-\theta_1)} = r_2 e^{i\theta} e^{-i\theta_1} = \frac{r_1 r_2 e^{i\theta}}{r_1 e^{i\theta_1}} = \frac{z_1z_2}{z_1} = z_2
\]

and so \( \theta_2 \in \arg(z_2) \). Therefore \( \theta_1 + \theta_2 = \theta \in \arg(z_1) + \arg(z_2) \). Similarly, if \( \theta_2 \in \arg(z_2) \) then \( \theta_1 = \theta - \theta_2 \in \arg(z_1) \) and \( \theta_1 + \theta_2 = \theta \in \arg(z_1) + \arg(z_2) \). That is, \( \arg(z_1z_2) \subset \arg(z_1) + \arg(z_2) \). Hence, \( \arg(z_1z_2) = \arg(z_1) + \arg(z_2) \).
Lemma 1.8.1. For any \( z_1, z_2 \in \mathbb{C} \) with \( z_1 \neq 0, z_2 \neq 0 \), we have \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \).

**Proof.** Let \( \theta_1 + \theta_2 \in \arg(z_1) + \arg(z_2) \) (so that \( z_1 = r_1 e^{i \theta_1} \) and \( z_2 = r_2 e^{i \theta_2} \)). Then \( z_1 z_2 = (r_1 e^{i \theta_1})(r_2 e^{i \theta_2}) = (r_1 r_2)e^{i(\theta_1 + \theta_2)} \) and so \( \theta_1 + \theta_2 \in \arg(z_1 z_2) \). So \( \arg(z_1) + \arg(z_2) \subset \arg(z_1 z_2) \).

Now let \( \theta \in \arg(z_1 z_2) \). Then \( z_1 z_2 = r_1 r_2 e^{i \theta} \) where \( r_1 = |z_1| \) and \( r_2 = |z_2| \).

Let \( \theta_1 \in \arg(z_1) \). Then \( z_1 = r_1 e^{i \theta_1} \). Let \( \theta_2 = \theta - \theta_1 \). Then

\[
  r_2 e^{i \theta_2} = r_2 e^{i(\theta - \theta_1)} = r_2 e^{i \theta} e^{-i \theta_1} = \frac{r_1 r_2 e^{i \theta}}{r_1 e^{i \theta_1}} = \frac{z_1 z_2}{z_1} = z_2
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and so \( \theta_2 \in \arg(z_2) \). Therefore \( \theta_1 + \theta_2 = \theta \in \arg(z_1) + \arg(z_2) \). Similarly, if \( \theta_2 \in \arg(z_2) \) then \( \theta_1 = \theta - \theta_2 \in \arg(z_1) \) and

\( \theta_1 + \theta_2 = \theta \in \arg(z_1) + \arg(z_2) \). That is, \( \arg(z_1 z_2) \subset \arg(z_1) + \arg(z_2) \).

Hence, \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \). \( \square \)
Lemma 1.8.2

**Lemma 1.8.2.** For any \( z_1, z_2 \in \mathbb{C} \) with \( z_1 \neq 0, z_2 \neq 0 \) we have \( \arg(z_1/z_2) = \arg(z_1) - \arg(z_2) \).

**Proof.** First, we have for \( z_2 = r_2 e^{i\theta_2} \) that \( z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2} \) (by Theorem 1.7.1), so \( \arg(z_2^{-1}) = -\arg(z_2) \). Now by Lemma 1.8.1,

\[
\arg(z_1/z_2) = \arg(z_1z_2^{-1}) = \arg(z_1) + \arg(z_2^{-1}) = \arg(z_1) - \arg(z_2).
\]
Lemma 1.8.2. For any $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq 0$, $z_2 \neq 0$ we have
\[ \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2). \]

Proof. First, we have for $z_2 = r_2 e^{i\theta_2}$ that $z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2}$ (by Theorem 1.7.1), so $\arg(z_2^{-1}) = -\arg(z_2)$. Now by Lemma 1.8.1,
\[
\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1 z_2^{-1}) = \arg(z_1) + \arg(z_2^{-1}) = \arg(z_1) - \arg(z_2).
\]