Lemma 2.15.A

Lemma 2.15.A. Let \( f \) be a function defined at all points \( z \) in some deleted neighborhood of \( z_0 \). If \( \lim_{z \to z_0} f(z) = w_0 \) and \( \lim_{z \to z_0} f(z) = w_1 \), then \( w_0 = w_1 \).

Proof. Let \( \varepsilon > 0 \). Then there are \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that
\[
0 < |z - z_0| < \delta_1 \implies |f(z) - w_0| < \varepsilon/2, \quad \text{and} \quad 0 < |z - z_0| < \delta_2 \implies |f(z) - w_1| < \varepsilon/2.
\]
Let \( \delta = \min\{\delta_1, \delta_2\} \). Then \( \delta > 0 \) and if
\[
0 < |z - z_0| < \delta \text{ then}
\]
\[
|w_1 - w_0| = |(f(z) - w_0) - (f(z) - w_1)| \\
\leq |f(z) - w_0| + |f(z) - w_1| \text{ by the Triangle Inequality} \\
< \varepsilon/2 + \varepsilon/2 = \varepsilon
\]
Since \( \varepsilon > 0 \) can be arbitrarily small, then it must be that \( w_0 = w_1 \), as claimed. \( \square \)