Chapter 2. Analytic Functions
Section 2.17. Limits Involving the Point at Infinity—Proofs of Theorems
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\begin{align*}
\lim_{z \to z_0} f(z) &= \infty \quad \text{if and only if} \quad \lim_{z \to z_0} \frac{1}{f(z)} = 0 \\
\lim_{z \to \infty} f(z) &= w_0 \quad \text{if and only if} \quad \lim_{z \to 0} f(1/z) = w_0, \quad \text{and} \\
\lim_{z \to \infty} f(z) &= \infty \quad \text{if and only if} \quad \lim_{z \to 0} \frac{1}{f(1/z)} = 0.
\end{align*}
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Proof. Let \( \varepsilon > 0 \) and define \( g(z) = 1/f(z) \), \( h(z) = f(1/z) \), and \( k(z) = 1/f(1/z) \).
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Proof. Let $\varepsilon > 0$ and define $g(z) = 1/f(z)$, $h(z) = f(1/z)$, and $k(z) = 1/f(1/z)$.

First, suppose $\lim_{z \to z_0} f(z) = \infty$. Then (by definition) there exists $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $1/|f(z)| < \varepsilon$. So $0 < |z - z_0| < \delta$ implies $1/|f(z)| = |g(z) - 0| < \varepsilon$. 
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$\lim_{z \to z_0} g(z) = \lim_{z \to z_0} 1/f(z) = 0$. 
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\[
\lim_{z \to z_0} g(z) = \lim_{z \to z_0} 1/f(z) = 0.
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Next, suppose \( \lim_{z \to z_0} 1/f(z) = 0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 0 < |z - z_0| < \delta \) implies \( |1/f(z) - 0| < \varepsilon \). So \( 0 < |z - z_0| < \delta \) implies \( |1/f(z) - 0| = 1/|f(z)| < \varepsilon \).
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Proof. Let \( \varepsilon > 0 \) and define \( g(z) = 1/f(z) \), \( h(z) = f(1/z) \), and \( k(z) = 1/f(1/z) \).

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\lim_{z \to z_0} g(z) = \lim_{z \to z_0} 1/f(z) = 0.
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Next, suppose \( \lim_{z \to z_0} 1/f(z) = 0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 0 < |z - z_0| < \delta \) implies \( |1/f(z) - 0| < \varepsilon \). So \( 0 < |z - z_0| < \delta \) implies \( |1/f(z) - 0| = 1/|f(z)| < \varepsilon \). Therefore (by definition)
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Proof. Let $\varepsilon > 0$ and define $g(z) = 1/f(z)$, $h(z) = f(1/z)$, and $k(z) = 1/f(1/z)$.

First, suppose $\lim_{z \to z_0} f(z) = \infty$. Then (by definition) there exists $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $1/|f(z)| < \varepsilon$. So $0 < |z - z_0| < \delta$ implies $1/|f(z)| = |g(z) - 0| < \varepsilon$. Therefore (by definition)

$\lim_{z \to z_0} g(z) = \lim_{z \to z_0} 1/f(z) = 0$.

Next, suppose $\lim_{z \to z_0} 1/f(z) = 0$. Then (by definition) there exists $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $|1/f(z) - 0| < \varepsilon$. So $0 < |z - z_0| < \delta$ implies $|1/f(z) - 0| = 1/|f(z)| < \varepsilon$. Therefore (by definition)

$\lim_{z \to z_0} f(z) = \infty.$
Theorem 2.17.1 (continued 1)

**Proof (continued).** Second, suppose \( \lim_{z \to \infty} f(z) = w_0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 1/|z| < \delta \) implies \( |f(z) - w_0| < \varepsilon \). So (replacing \( z \) with \( 1/z \)) \( 0 < |z| = |z - 0| < \delta \) implies \( |f(1/z) - w_0| = |h(z) - w_0| < \varepsilon \). Therefore (by definition) \( \lim_{z \to 0} h(z) = \lim_{z \to 0} f(1/z) = w_0 \).
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**Proof (continued).** Second, suppose \( \lim_{z \to \infty} f(z) = w_0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 1/|z| < \delta \) implies \( |f(z) - w_0| < \varepsilon \). So (replacing \( z \) with \( 1/z \)) \( 0 < |z| = |z - 0| < \delta \) implies \( |f(1/z) - w_0| = |h(z) - w_0| < \varepsilon \). Therefore (by definition) \( \lim_{z \to 0} h(z) = \lim_{z \to 0} f(1/z) = w_0 \).

Suppose \( \lim_{z \to 0} f(1/z) = w_0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 0 < |z - 0| < \delta \) implies \( |f(1/z) - w_0| < \varepsilon \). So (replacing \( z \) with \( 1/z \)) \( 0 < |1/z - 0| = 1/|z| < \delta \) implies \( |f(z) - w_0| < \varepsilon \).
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So (replacing \( z \) with \( 1/z \)) \( 0 < |z| = |z - 0| < \delta \) implies \( |f(1/z) - w_0| = |h(z) - w_0| < \varepsilon \). Therefore (by definition)
\[
\lim_{z \to 0} h(z) = \lim_{z \to 0} f(1/z) = w_0.
\]

Suppose \( \lim_{z \to 0} f(1/z) = w_0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 0 < |z - 0| < \delta \) implies \( |f(1/z) - w_0| < \varepsilon \). So (replacing \( z \) with \( 1/z \))
\[
0 < |1/z - 0| = 1/|z| < \delta \text{ implies } |f(z) - w_0| < \varepsilon.
\]

Therefore (by definition)
\[
\lim_{z \to \infty} f(z) = w_0.
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Proof (continued). Second, suppose $\lim_{z \to \infty} f(z) = w_0$. Then (by definition) there exists $\delta > 0$ such that $1/|z| < \delta$ implies $|f(z) - w_0| < \varepsilon$. So (replacing $z$ with $1/z$) $0 < |z| = |z - 0| < \delta$ implies $|f(1/z) - w_0| = |h(z) - w_0| < \varepsilon$. Therefore (by definition) $\lim_{z \to 0} h(z) = \lim_{z \to 0} f(1/z) = w_0$.

Suppose $\lim_{z \to 0} f(1/z) = w_0$. Then (by definition) there exists $\delta > 0$ such that $0 < |z - 0| < \delta$ implies $|f(1/z) - w_0| < \varepsilon$. So (replacing $z$ with $1/z$) $0 < |1/z - 0| = 1/|z| < \delta$ implies $|f(z) - w_0| < \varepsilon$. Therefore (by definition) $\lim_{z \to \infty} f(z) = w_0$.

Third, suppose $\lim_{z \to \infty} f(z) = \infty$. Then (by definition) there exists $\delta > 0$ such that $1/|z| < \delta$ implies $1/|f(z)| < \varepsilon$. So (replacing $z$ with $1/z$) $0 < |z| < \delta$ implies $|1/f(1/z)| < \varepsilon$. So $0 < |z - 0| < \delta$ implies $|k(z) - 0| < \varepsilon$. 
Theorem 2.17.1 (continued 1)

Proof (continued). Second, suppose \( \lim_{z \to \infty} f(z) = w_0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 1/|z| < \delta \) implies \( |f(z) - w_0| < \varepsilon \). So (replacing \( z \) with \( 1/z \)) \( 0 < |z| = |z - 0| < \delta \) implies \( |f(1/z) - w_0| = |h(z) - w_0| < \varepsilon \). Therefore (by definition) \( \lim_{z \to 0} h(z) = \lim_{z \to 0} f(1/z) = w_0 \).

Suppose \( \lim_{z \to 0} f(1/z) = w_0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 0 < |z - 0| < \delta \) implies \( |f(1/z) - w_0| < \varepsilon \). So (replacing \( z \) with \( 1/z \)) \( 0 < |1/z - 0| = 1/|z| < \delta \) implies \( |f(z) - w_0| < \varepsilon \). Therefore (by definition) \( \lim_{z \to \infty} f(z) = w_0 \).

Third, suppose \( \lim_{z \to \infty} f(z) = \infty \). Then (by definition) there exists \( \delta > 0 \) such that \( 1/|z| < \delta \) implies \( 1/|f(z)| < \varepsilon \). So (replacing \( z \) with \( 1/z \)) \( 0 < |z| < \delta \) implies \( |1/f(1/z)| < \varepsilon \). So \( 0 < |z - 0| < \delta \) implies \( |k(z) - 0| < \varepsilon \). Therefore (by definition) \( \lim_{z \to 0} k(z) = \lim_{z \to 0} 1/f(1/z) = 0 \).
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Proof (continued). Second, suppose \( \lim_{z \to \infty} f(z) = w_0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 1/|z| < \delta \) implies \( |f(z) - w_0| < \varepsilon \).

So (replacing \( z \) with \( 1/z \)) \( 0 < |z| = |z - 0| < \delta \) implies \( |f(1/z) - w_0| = |h(z) - w_0| < \varepsilon \). Therefore (by definition) \( \lim_{z \to 0} h(z) = \lim_{z \to 0} f(1/z) = w_0 \).

Suppose \( \lim_{z \to 0} f(1/z) = w_0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 0 < |z - 0| < \delta \) implies \( |f(1/z) - w_0| < \varepsilon \). So (replacing \( z \) with \( 1/z \)) \( 0 < |1/z - 0| = 1/|z| < \delta \) implies \( |f(z) - w_0| < \varepsilon \). Therefore (by definition) \( \lim_{z \to \infty} f(z) = w_0 \).

Third, suppose \( \lim_{z \to \infty} f(z) = \infty \). Then (by definition) there exists \( \delta > 0 \) such that \( 1/|z| < \delta \) implies \( 1/|f(z)| < \varepsilon \). So (replacing \( z \) with \( 1/z \)) \( 0 < |z| < \delta \) implies \( |1/f(1/z)| < \varepsilon \). So \( 0 < |z - 0| < \delta \) implies \( |k(z) - 0| < \varepsilon \). Therefore (by definition) \( \lim_{z \to 0} k(z) = \lim_{z \to 0} 1/f(1/z) = 0 \).
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\]

Proof (continued). Suppose $\lim_{z \to 0} \frac{1}{f(1/z)} = 0$. Then (by definition) there exists $\delta > 0$ such that $0 < |z - 0| < \delta$ implies $|1/f(1/z) - 0| < \varepsilon$. So (replacing $z$ with $1/z$) $0 < |1/z| < \delta$ implies $|1/f(z)| < \varepsilon$. Therefore (by definition) $\lim_{z \to \infty} f(z) = \infty$. \qed
Theorem 2.17.1 (continued 2)

Theorem 2.17.1. If \( z_0, w_0 \in \mathbb{C} \) then

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\begin{align*}
\lim_{z \to z_0} f(z) &= \infty \quad \text{if and only if} \quad \lim_{z \to z_0} \frac{1}{f(z)} = 0 \\
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\end{align*}
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Proof (continued). Suppose \( \lim_{z \to 0} \frac{1}{f(1/z)} = 0 \). Then (by definition) there exists \( \delta > 0 \) such that \( 0 < |z - 0| < \delta \) implies \( |1/f(1/z) - 0| < \varepsilon \). So (replacing \( z \) with \( 1/z \)) \( 0 < |1/z| < \delta \) implies \( |1/f(z)| < \varepsilon \). Therefore (by definition) \( \lim_{z \to \infty} f(z) = \infty \). \( \square \)