Chapter 2. Analytic Functions
Section 2.20. Differentiation Formulas—Proofs of Theorems
Theorem 2.20.A

Let \( c \in \mathbb{C} \) and let \( f \) and \( g \) be functions where derivatives exist at a point \( z \in \mathbb{C} \). Then:

\[
\frac{d}{dz}[c] = 0, \quad \frac{d}{dz}[z] = 1, \quad \frac{d}{dz}[cf(z)] = c \frac{d}{dz}[f],
\]

and

\[
\frac{d}{dz}[f(z) + g(z)] = \frac{d}{dz}[f(z)] + \frac{d}{dz}[g(z)].
\]

Proof. By the definition of derivative:

\[
\frac{d}{dz}[c] = \lim_{\Delta z \to 0} \frac{(c) - (c)}{\Delta z} = 0
\]

\[
\frac{d}{dz}[z] = \lim_{\Delta z \to 0} \frac{(z + \Delta z) - z}{\Delta z} = 1
\]

\[
\frac{d}{dz}[cf(z)] = \lim_{\Delta z \to 0} \frac{cf(z + \Delta z) - cf(z)}{\Delta z}
\]

\[
= c \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = c \frac{d}{dz}[f(z)]
\]
Theorem 2.20.A

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\frac{d}{dz}[c] = 0, \quad \frac{d}{dz}[z] = 1, \quad \frac{d}{dz}[cf(z)] = c \frac{d}{dz}[f],
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and

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\frac{d}{dz}[f(z) + g(z)] = \frac{d}{dz}[f(z)] + \frac{d}{dz}[g(z)].
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Proof. By the definition of derivative:

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\frac{d}{dz}[z] = \lim_{\Delta z \to 0} \frac{(z + \Delta z) - z}{\Delta z} = 1
$$

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\frac{d}{dz}[cf(z)] = \lim_{\Delta z \to 0} \frac{cf(z + \Delta z) - cf(z)}{\Delta z}
$$

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= c \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = c \frac{d}{dz}[f(z)]
$$
Theorem 2.20.A (continued)

**Theorem 2.20.A.** Let \( c \in \mathbb{C} \) and let \( f \) and \( g \) be functions where derivatives exist at a point \( z \in \mathbb{C} \). Then:

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\frac{d}{dz}[c] = 0, \quad \frac{d}{dz}[z] = 1, \quad \frac{d}{dz}[cf(z)] = c \frac{d}{dz}[f],
\]

and

\[
\frac{d}{dz}[f(z) + g(z)] = \frac{d}{dz}[f(z)] + \frac{d}{dz}[g(z)].
\]

**Proof (continued).** By the definition of derivative:

\[
\frac{d}{dz}[f(z) + g(z)] = \lim_{\Delta z \to 0} \frac{(f(z + \Delta z) + g(z + \Delta z) - (f(z) + g(z))}{\Delta z}
\]

\[
= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z}
\]

\[
= \frac{d}{dz}[f(z)] + \frac{d}{dz}[g(z)]
\]
Theorem 2.20.B

**Theorem 2.20.B.** If $f$ and $g$ are functions whose derivative exists at a point $z$ then

Product Rule: \[
\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)
\]

Quotient Rule: \[
\frac{d}{dz} \left[ \frac{f(z)}{g(z)} \right] = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2} \quad \text{if } g(z) \neq 0.
\]

**Proof.** By the definition of derivative, \[
\frac{d}{dz}[f(z)g(z)] = \lim_{\Delta z \to 0} \frac{f(z + \Delta z)g(z + \Delta z) - f(z)g(z)}{\Delta z}
\]

\[
= \lim_{\Delta z \to 0} \left( \frac{(f(z + \Delta z)g(z + \Delta z) - f(z)g(z + \Delta z))}{\Delta z} + \frac{f(z)(g(z + \Delta z) - g(z))}{\Delta z} \right)
\]

\[
= \lim_{\Delta z \to 0} \left( \frac{(f(z + \Delta z) - f(z))g(z + \Delta z)}{\Delta z} + \frac{f(z)(g(z + \Delta z) - g(z))}{\Delta z} \right)
\]

\[
= \lim_{\Delta z \to 0} \left( f'(z)g(z) + f(z)g'(z) \right)
\]

\[
= f'(z)g(z) + f(z)g'(z)
\]
Theorem 2.20.B

Theorem 2.20.B. If $f$ and $g$ are functions whose derivative exists at a point $z$ then

- **Product Rule:** $\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$ and
- **Quotient Rule:** $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$ if $g(z) \neq 0$.

Proof. By the definition of derivative, $\frac{d}{dz}[f(z)g(z)] =$

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z)g(z + \Delta z) - f(z)g(z)}{\Delta z} = \lim_{\Delta z \to 0} \left( (f(z + \Delta z)g(z + \Delta z) - f(z)g(z + \Delta z) + f(z)g(z + \Delta z) - f(z)g(z)) / \Delta z \right)$$
Theorem 2.20.B (continued)

Proof (continued).

\[
= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \lim_{\Delta z \to 0} g(z + \Delta z) \\
+ \lim_{\Delta z \to 0} f(z) \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} \\
= f'(z) \lim_{\Delta z \to 0} g(z + \Delta z) + f(z)g'(z) \\
= f'(z)g(z) + f(z)g'(z) \text{ since } g \text{ is continuous at } z \text{ by Theorem 2.19.A.}
\]

By the definition of derivative, \( \frac{d}{dz} \left[ \frac{f(z)}{g(z)} \right] = \)

\[
\frac{d}{dz} \left[ \frac{f(z)}{g(z)} \right] = \lim_{\Delta z \to 0} \frac{\frac{f(z + \Delta z)}{g(z + \Delta z)} - \frac{f(z)}{g(z)}}{\Delta z} \\
= \lim_{\Delta z \to 0} \frac{g(z)f(z + \Delta z) - f(z)g(z + \Delta z)}{\Delta z g(z + \Delta z)g(z)}
\]
Theorem 2.20.B (continued)

Proof (continued).

\[
= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \lim_{\Delta z \to 0} g(z + \Delta z) \\
+ \lim_{\Delta z \to 0} f(z) \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z}
\]

\[
= f'(z) \lim_{\Delta z \to 0} g(z + \Delta z) + f(z)g'(z)
\]

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By the definition of derivative, \( \frac{d}{dz} \left[ \frac{f(z)}{g(z)} \right] = \)

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= \lim_{\Delta z \to 0} \frac{g(z)f(z + \Delta z) - f(z)g(z + \Delta z)}{\Delta z g(z + \Delta z)g(z)}
\]
Theorem 2.20.B (continued)

Proof (continued).

\[
\begin{align*}
&= \lim_{\Delta z \to 0} \frac{g(z)f(z + \Delta z) - g(z)f(z) + g(z)f(z) - f(z)g(z + \Delta z)}{\Delta z g(z + \Delta z) g(z)} \\
&= \lim_{\Delta z \to 0} \frac{g(z) \frac{f(z + \Delta z) - f(z)}{\Delta z} - f(z) \frac{g(z + \Delta z) - g(z)}{\Delta z}}{g(z + \Delta z) g(z)} \\
&= \lim_{\Delta z \to 0} g(z) \frac{f(z + \Delta z) - f(z)}{\Delta z} - \lim_{\Delta z \to 0} f(z) \frac{g(z + \Delta z) - g(z)}{\Delta z} \\
&= g(z) \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} - f(z) \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} \\
&= g(z) \frac{f'(z)}{g(z)} - f(z) \frac{g'(z)}{g(z)} \\
&= \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.
\end{align*}
\]
Corollary 2.20.A

**Corollary 2.20.A.** If \( n \in \mathbb{N} \) then \( \frac{d}{dz}[z^n] = nz^{n-1} \).

**Proof.** The traditional way to prove this is using the Binomial Theorem (Theorem 1.3.2) and you are asked to do this in Exercise 2.20.6b. Here we give an inductive proof (this is Exercise 2.20.6a).
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We have the result $\frac{d}{dz}[z^1] = 1$ by Theorem 2.20.A so the base case $n = 1$ holds.
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We have the result $\frac{d}{dz}[z^1] = 1$ by Theorem 2.20.A so the base case $n = 1$ holds. Now assume the result holds for $n = k$ and consider $n = k + 1$:

$$
\frac{d}{dz}[z^n] = \frac{d}{dz}[z^{k+1}] = \frac{d}{dz}[z^k z] = [kz^{k-1}](z) + (z^k)[1] \text{ by the Product Rule (Theorem 2.20.B)}
$$

and the induction hypothesis

$$
= kz^k + z^k = (k + 1)z^k.
$$

Therefore the claim holds for $n = k + 1$ and hence by Mathematical Induction it holds for all $n \in \mathbb{N}$. 

\[\square\]
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\frac{d}{dz}[z^n] = \frac{d}{dz}[z^{k+1}] = \frac{d}{dz}[z^k z]
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= [kz^{k-1}](z) + (z^k)[1] \text{ by the Product Rule (Theorem 2.20.B)}
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Therefore the claim holds for \( n = k + 1 \) and hence by Mathematical Induction it holds for all \( n \in \mathbb{N} \). \( \square \)
Theorem 2.20.C. The Chain Rule.

Suppose that $f$ has a derivative at $z_0$ and that $g$ has a derivative at $f(z_0)$. Then the composition function $F(z) = (g \circ f)(z) = g(f(z))$ has a derivative at $z_0$ and $F'(z_0) = g'(f(z_0))f'(z_0)$. In differential notation with $w = f(z)$ and $W = g(w)$ (so that $W = F(z)$) we have $\frac{dW}{dz} = \frac{dW}{dw} \frac{dw}{dz}$.

**Proof.** Denote $w_0 = f(z_0)$. Since $g'(w_0) = g'(f(z_0))$ exists by hypothesis then by the definition of derivative, there is some $\varepsilon > 0$ such that $g$ is defined on $|w - w_0| < \varepsilon$. 

Theorem 2.20.C. The Chain Rule.

Suppose that $f$ has a derivative at $z_0$ and that $g$ has a derivative at $f(z_0)$. Then the composition function $F(z) = (g \circ f)(z) = g(f(z))$ has a derivative at $z_0$ and $F'(z_0) = g'(f(z_0))f'(z_0)$. In differential notation with $w = f(z)$ and $W = g(w)$ (so that $W = F(z)$) we have $\frac{dW}{dz} = \frac{dW}{dw} \frac{dw}{dz}$.

Proof. Denote $w_0 = f(z_0)$. Since $g'(w_0) = g'(f(z_0))$ exists by hypothesis then by the definition of derivative, there is some $\varepsilon > 0$ such that $g$ is defined on $|w - w_0| < \varepsilon$. So we define

$$\Phi(w) = \begin{cases} \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) & \text{if } w \neq w_0 \\ 0 & \text{if } w = w_0 \end{cases}$$
Theorem 2.20.C. The Chain Rule.
Suppose that $f$ has a derivative at $z_0$ and that $g$ has a derivative at $f(z_0)$. Then the composition function $F(z) = (g \circ f)(z) = g(f(z))$ has a derivative at $z_0$ and $F'(z_0) = g'(f(z_0))f'(z_0)$. In differential notation with $w = f(z)$ and $W = g(w)$ (so that $W = F(z)$) we have $\frac{dW}{dz} = \frac{dW}{dw} \frac{dw}{dz}$.

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0 & \text{if } w = w_0
\end{cases}
$$
Theorem 2.20.C (continued 1)

Proof (continued) . . . and note that

\[ \lim_{w \to w_0} \Phi(w) = \lim_{w \to w_0} \left( \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \right) \]

\[ = \lim_{w \to w_0} \left( \frac{g(w) - g(w_0)}{w - w_0} \right) - g'(w_0) = g'(w_0) - g'(w_0) = 0 = \Phi(w_0). \]

So \( \Phi \) is continuous at \( w_0 \). Rearranging the definition of \( \Phi \) we get

\[ g(w) - g(w_0) = (g'(w_0) + \Phi(w))(w - w_0) \text{ for } |w - w_0| < \varepsilon. \]

Since \( f'(z_0) \) exists then \( f \) is continuous at \( z_0 \) by Theorem 2.19.A, so by definition of continuity there is \( \delta > 0 \) such that for all \( |z - z_0| < \delta \) we have

\[ |f(z) - f(z_0)| = |w - w_0| < \varepsilon. \]
Proof (continued). ... and note that

\[
\lim_{w \to w_0} \Phi(w) = \lim_{w \to w_0} \left( \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \right)
\]

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= \lim_{w \to w_0} \left( \frac{g(w) - g(w_0)}{w - w_0} \right) - g'(w_0) = g'(w_0) - g'(w_0) = 0 = \Phi(w_0).
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\[
g(f(z)) - g(f(z_0)) = (g'(f(z_0)) + \Phi(f(z)))(f(z) - f(z_0)) \text{ for } |z - z_0| < \delta.
\]
Theorem 2.20.C (continued 1)

Proof (continued). ...and note that

\[
\lim_{w \to w_0} \Phi(w) = \lim_{w \to w_0} \left( \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \right)
\]

\[
= \lim_{w \to w_0} \left( \frac{g(w) - g(w_0)}{w - w_0} \right) - g'(w_0) = g'(w_0) - g'(w_0) = 0 = \Phi(w_0).
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So \( \Phi \) is continuous at \( w_0 \). Rearranging the definition of \( \Phi \) we get

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\[
g(f(z)) - g(f(z_0)) = (g'(f(z_0)) + \Phi(f(z)))(f(z) - f(z_0)) \text{ for } |z - z_0| < \delta
\]
Proof (continued). ... or
\[
\frac{g(f(z)) - g(f(z_0))}{z - z_0} = (g'(f(z_0)) + \Phi(f(z))) \frac{f(z) - f(z_0)}{z - z_0}
\]
for \(0 < |z - z_0| < \delta\).

Therefore
\[
\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \lim_{z \to z_0} (g'(f(z_0)) + \Phi(f(z))) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]
or
\[
d \frac{d}{dz} \left| \frac{g(f(z)) - g(f(z_0))}{z - z_0} \right| \bigg|_{z = z_0} = g'(f(z_0)) f'(z_0) + \Phi(f(z_0)) f'(z_0)
\]
since \(f\) is continuous at \(z_0\) and \(\Phi\) is continuous at \(w_0 = f(z_0)\).
Theorem 2.20.C (continued 2)

Proof (continued). . . . or

\[
g(f(z)) - g(f(z_0)) = (g'(f(z_0)) + \Phi(f(z))) \frac{f(z) - f(z_0)}{z - z_0} \text{ for } 0 < |z - z_0| < \delta.
\]

Therefore

\[
\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \lim_{z \to z_0} \left( g'(f(z_0)) + \Phi(f(z)) \right) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]

or

\[
\frac{d}{dz}[g(f(z))]|_{z=z_0} = g'(f(z_0))f'(z_0) + \lim_{z \to z_0} \Phi(f(z))f'(z_0)
\]

\[
= g'(f(z_0))f'(z_0) + \Phi(f(z_0))f'(z_0) \text{ since } f \text{ is continuous at } z_0 \text{ and } \Phi \text{ is continuous at } w_0 = f(z_0)
\]

\[
= g'(f(z_0))f'(z_0) \text{ since } \Phi(f(z_0)) = \Phi(w_0) = 0.
\]
Theorem 2.20.C (continued 2)

**Proof (continued).** ... or

\[
\frac{g(f(z)) - g(f(z_0))}{z - z_0} = (g'(f(z_0)) + \Phi(f(z))) \frac{f(z) - f(z_0)}{z - z_0}
\]

for \(0 < |z - z_0| < \delta\).

Therefore

\[
\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \lim_{z \to z_0} (g'(f(z_0)) + \Phi(f(z))) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
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or

\[
\frac{d}{dz}[g(f(z))] \bigg|_{z=z_0} = g'(f(z_0))f'(z_0) + \lim_{z \to z_0} \Phi(f(z))f'(z_0)
\]

\[
= g'(f(z_0))f'(z_0) + \Phi(f(z_0))f'(z_0) \text{ since } f \text{ is continuous at } z_0 \text{ and } \Phi \text{ is continuous at } w_0 = f(z_0)
\]

\[
= g'(f(z_0))f'(z_0) \text{ since } \Phi(f(z_0)) = \Phi(w_0) = 0.
\]