Chapter 2. Analytic Functions

Section 2.21. Cauchy-Riemann Equations—Proofs of Theorems
Theorem 2.21.A. Differentiable Implies the C-R Equations
Theorem 2.21.A

Theorem 2.21.A. Differentiable Implies the Cauchy-Riemann Equations
Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of $u$ and $v$ must exist at $(x_0, y_0)$, and they must satisfy the Cauchy-Riemann equations:

$$\frac{\partial}{\partial x}[u(x, y)] = \frac{\partial}{\partial y}[v(x, y)] \quad \text{and} \quad \frac{\partial}{\partial y}[u(x, y)] = -\frac{\partial}{\partial x}[v(x, y)]$$

(or with subscripts representing partial derivatives, $u_x = v_y$ and $u_y = -v_x$) at $(x_0, y_0)$. Also, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Proof. Suppose $f'$ exists at $z_0 = x_0 + iy_0$. Let $\Delta z = \Delta x + i\Delta y$. Then with $w = f(z)$ we have:
Theorem 2.21.A. Differentiable Implies the Cauchy-Riemann Equations

Suppose that \( f(z) = u(x, y) + iv(x, y) \) and that \( f' \) exists at a point \( z_0 = x_0 + iy_0 \). Then the first-order partial derivatives of \( u \) and \( v \) must exist at \( (x_0, y_0) \), and they must satisfy the Cauchy-Riemann equations:

\[
\frac{\partial}{\partial x} [u(x, y)] = \frac{\partial}{\partial y} [v(x, y)] \quad \text{and} \quad \frac{\partial}{\partial y} [u(x, y)] = -\frac{\partial}{\partial x} [v(x, y)]
\]

(or with subscripts representing partial derivatives, \( u_x = v_y \) and \( u_y = -v_x \)) at \( (x_0, y_0) \). Also, \( f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \).

**Proof.** Suppose \( f' \) exists at \( z_0 = x_0 + iy_0 \). Let \( \Delta z = \Delta x + i\Delta y \). Then with \( w = f(z) \) we have:
Theorem 2.21.A (continued 1)

Proof (continued).

\[
\frac{\Delta w}{\Delta z} = \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}
\]

Then \(f'(z_0) = \lim_{\Delta z \to 0}(\Delta w/\Delta z)\), so by Theorem 2.16.1,

\[
f'(z_0) = \lim_{(\Delta x, \Delta y) \to (0,0)} \text{Re}\left(\frac{\Delta w}{\Delta z}\right) + i \lim_{(\Delta x, \Delta y) \to (0,0)} \text{Im}\left(\frac{\Delta w}{\Delta z}\right). \quad (3)
\]

We now apply the contrapositive of the Two-Path Test for the Nonexistence of a Limit for a function of two variables (see Note 2.15.A), which implies that if a limit exists as \(\Delta z \to 0\) then the limit exists and is the same along all paths for which \(\Delta z \to 0\).
Theorem 2.21.A (continued 1)

Proof (continued).

\[
\frac{\Delta w}{\Delta z} = \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)\} + i\{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)\}}{\Delta x + i\Delta y}
\]

Then \(f'(z_0) = \lim_{\Delta z \to 0}(\Delta w/\Delta z)\), so by Theorem 2.16.1,

\[
f'(z_0) = \lim_{(\Delta x, \Delta y) \to (0,0)} \text{Re} \left( \frac{\Delta w}{\Delta z} \right) + i \lim_{(\Delta x, \Delta y) \to (0,0)} \text{Im} \left( \frac{\Delta w}{\Delta z} \right).
\]  \hspace{1cm} (3)

We now apply the contrapositive of the Two-Path Test for the Nonexistence of a Limit for a function of two variables (see Note 2.15.A), which implies that if a limit exists as \(\Delta z \to 0\) then the limit exists and is the same along all paths for which \(\Delta z \to 0\).
Theorem 2.21.A (continued 2)

**Proof (continued).** In particular, we can let $\Delta z \to 0$ along the real axis (where $\Delta y = 0$) or along the imaginary axis (where $\Delta x = 0$).

We first consider $\Delta z \to 0$ along the real axis and have for the real and imaginary parts of equation (3) that

$$
\lim_{(\Delta x, \Delta y) \to (0,0)} \text{Re} \left( \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}
$$

$$
= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = u_x(x_0, y_0)
$$
Theorem 2.21.A (continued 2)

Proof (continued). In particular, we can let $\Delta z \to 0$ along the real axis (where $\Delta y = 0$) or along the imaginary axis (where $\Delta x = 0$). We first consider $\Delta z \to 0$ along the real axis and have for the real and imaginary parts of equation (3) that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \text{Re} \left( \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = u_x(x_0, y_0)$$

and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \text{Im} \left( \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = v_x(x_0, y_0).$$
Theorem 2.21.A (continued 2)

**Proof (continued).** In particular, we can let \( \Delta z \to 0 \) along the real axis (where \( \Delta y = 0 \)) or along the imaginary axis (where \( \Delta x = 0 \)). We first consider \( \Delta z \to 0 \) along the real axis and have for the real and imaginary parts of equation (3) that

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and

\[
\lim_{(\Delta x, \Delta y) \to (0,0)} \text{Im} \left( \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = v_x(x_0, y_0).
\]

So by (3), \( f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \) and \( f' \) has the form as claimed.
Proof (continued). In particular, we can let $\Delta z \to 0$ along the real axis (where $\Delta y = 0$) or along the imaginary axis (where $\Delta x = 0$). We first consider $\Delta z \to 0$ along the real axis and have for the real and imaginary parts of equation (3) that

$$\lim_{(\Delta x, \Delta y) \to (0, 0)} \text{Re} \left( \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = u_x(x_0, y_0)$$

and

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So by (3), $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$ and $f'$ has the form as claimed.
Theorem 2.21.A (continued 3)

Proof (continued). Second, with $\Delta z \to 0$ along the imaginary axis so that $\Delta z = i \Delta y$, $\Delta y \to 0$, and $\Delta x = 0$, we have

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i \Delta y} + i \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i \Delta y}$$

$$= \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}$$

and so

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \Re \left( \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} = v_y(x_0, y_0)$$

and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \Im \left( \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta y \to 0} - \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} = -u_y(x_0, y_0).$$
Proof (continued). Second, with $\Delta z \to 0$ along the imaginary axis so that $\Delta z = i \Delta y$, $\Delta y \to 0$, and $\Delta x = 0$, we have

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i \Delta y} + i \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i \Delta y}$$

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So by (3), $f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, y_0)$. 
Theorem 2.21.A (continued 3)

Proof (continued). Second, with $\Delta z \to 0$ along the imaginary axis so that $\Delta z = i\Delta y$, $\Delta y \to 0$, and $\Delta x = 0$, we have

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} + i \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$

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and so

$$\lim_{(\Delta x, \Delta y) \to (0, 0)} \text{Re} \left( \frac{\Delta w}{\Delta z} \right) = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} = v_y(x_0, y_0)$$

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So by (3), $f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$. 
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Suppose that \( f(z) = u(x, y) + iv(x, y) \) and that \( f' \) exists at a point \( z_0 = x_0 + iy_0 \). Then the first-order partial derivatives of \( u \) and \( v \) must exist at \( (x_0, y_0) \), and they must satisfy the Cauchy-Riemann equations:

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(or with subscripts representing partial derivatives, \( u_x = v_y \) and \( u_y = -v_x \)) at \( (x_0, y_0) \). Also, \( f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \).

Proof (continued). Since \( f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0) \), then we must have \( u_x(x_0, y_0) = v_y(x_0, y_0) \) and \( v_x(x_0, y_0) = -u_y(x_0, y_0) \).