Chapter 2. Analytic Functions
Section 2.22. Sufficient Conditions for Differentiability—Proofs of Theorems

Theorem 2.22.A

The Cauchy-Riemann Equations and Continuity Imply Differentiability
Let the function \( f(z) = u(x, y) + iv(x, y) \) be defined throughout some \( \varepsilon \) neighborhood of a point \( z_0 = x_0 + iy_0 \), and suppose that

(a) the first-order partial derivatives of the functions \( u \) and \( v \) with respect to \( x \) and \( y \) exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at \( (x_0, y_0) \) and satisfy the Cauchy-Riemann equations \( u_x(x_0, y_0) = v_y(x_0, y_0) \) and \( y_x(x_0, y_0) = -v_x(x_0, y_0) \).
Then \( f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \).

Proof. We present the proof given by Brown and Churchill. A more self-contained proof based on the Mean Value Theorem is given in my notes for Complex Analysis 1 (MATH 5510) on III.2. Analytic Functions.

Theorem 2.22.A (continued)

Proof (continued). Let \( \Delta z = \Delta x + i\Delta y \) where \( 0 < |\Delta z| < \varepsilon \) and let \( \Delta w = f(z_0 + \Delta z) - f(z_0) \). We take \( \Delta w = \Delta u + i\Delta v \) where

\[
\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0), \quad \Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).
\]

By (b), the first order partial derivatives of \( u \) and \( v \) are continuous at \( (x_0, y_0) \), so by a result from advanced calculus (see W. Kaplan’s Advanced Calculus, 5th ed., page 86 (2003)) we may write

\[
\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\]
\[
\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y
\]
where \( \varepsilon_i \to 0 \) as \( (\Delta x, \Delta y) \to (0, 0) \) for \( i = 1, 2, 3, 4 \). So we can express

\[
\Delta w = (u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y)
\]
\[
+ i(v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y).
\]

But \( |\Delta x| \leq |\Delta z| \) and \( |\Delta y| \leq |\Delta z| \) (by the Triangle Inequality, say), so

\[
|\Delta x/\Delta z| \leq 1 \quad \text{and} \quad |\Delta y/\Delta z| \leq 1.
\]
Proof (continued). Consequently,

\[ |(\varepsilon_1 + i\varepsilon_3) \frac{\Delta x}{\Delta z}| = |\varepsilon_1 + i\varepsilon_3| \left| \frac{\Delta x}{\Delta z} \right| \leq |\varepsilon_1 + i\varepsilon_3| \leq |\varepsilon_1| + |\varepsilon_3| \]

and

\[ |(\varepsilon_2 + i\varepsilon_4) \frac{\Delta x}{\Delta z}| = |\varepsilon_2 + i\varepsilon_4| \left| \frac{\Delta x}{\Delta z} \right| \leq |\varepsilon_2 + i\varepsilon_4| \leq |\varepsilon_2| + |\varepsilon_4|. \]

So as \( \Delta z = \Delta x + i\Delta y \rightarrow 0 \), we have that \(|(\varepsilon_1 + i\varepsilon_3) \Delta x / \Delta z| \rightarrow 0 \) and \(|(\varepsilon_2 + i\varepsilon_4) \Delta x / \Delta z| \rightarrow 0 \). Therefore,

\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\
= \lim_{\Delta z \to 0} \left( u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3) \frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4) \frac{\Delta y}{\Delta z} \right) \\
= u_x(x_0, y_0) + iv_x(x_0, y_0).
\]