Chapter 2. Analytic Functions
Section 2.23. Polar Coordinates—Proofs of Theorems
Lemma 2.23.A.
Lemma 2.23.A. Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some $\varepsilon$ neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

(a) the first-order partial derivatives of the functions $u$ and $v$ with respect to $x$ and $y$ exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at $(x_0, y_0)$ and satisfy the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $v_x(x_0, y_0) = -u_y(x_0, y_0)$.

Then with $z_0 = r_0 \exp(i\theta_0) \neq 0$ we have

$$r_0 u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \text{ and } u_\theta(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0).$$

These are the polar coordinate forms of the Cauchy-Riemann equations.

Proof. We have $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and, for $z \neq 0$, $z = r \exp(i\theta)$. Also, $x = r \cos \theta$ and $y = r \sin \theta$. 
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(a) the first-order partial derivatives of the functions \( u \) and \( v \) with respect to \( x \) and \( y \) exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at \( (x_0, y_0) \) and satisfy the Cauchy-Riemann equations

\[
\begin{align*}
    u_x(x_0, y_0) &= v_y(x_0, y_0) \\
    y_y(x_0, y_0) &= -v_x(x_0, y_0).
\end{align*}
\]

Then with \( z_0 = r_0 \exp(i\theta) \neq 0 \) we have

\[
    r_0 u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \quad \text{and} \quad u_\theta(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0).
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These are the polar coordinate forms of the Cauchy-Riemann equations.

Proof. We have \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) and, for \( z \neq 0 \), \( z = r \exp(i\theta) \). Also, \( x = r \cos \theta \) and \( y = r \sin \theta \).
Lemma 2.23.A (continued)

Proof (continued). By the Chain Rule

\[
\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = u_x \cos \theta + u_y \sin \theta \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = -u_x r \sin \theta + u_y r \cos \theta. \quad (2)
\]

Similarly,

\[
\frac{\partial v}{\partial r} = v_x \cos \theta + v_y \sin \theta \quad \text{and} \quad \frac{\partial v}{\partial \theta} = -v_x r \sin \theta + v_y r \cos \theta. \quad (3)
\]

Assuming the Cauchy-Riemann equations in \((x, y)\) hold, we have \(u_x = v_y\) and \(u_y = -v_x\) at \((x_0, y_0)\). So from (5)

\[
v_r = v_x \cos \theta + v_y \sin \theta = -u_y \cos \theta + u_x \sin \theta \quad \text{and} \quad v_\theta = -v_x r \sin \theta + v_y r \cos \theta = u_y r \sin \theta + u_x r \cos \theta \quad (5)
\]

at \((r_0, \theta_0)\).
Lemma 2.23.A (continued)

Proof (continued). By the Chain Rule

\[
\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = u_x \cos \theta + u_y \sin \theta \tag{2}
\]

and

\[
\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = -u_x r \sin \theta + u_y r \cos \theta.
\]

Similarly,

\[
\frac{\partial v}{\partial r} = v_x \cos \theta + v_y \sin \theta \tag{3}
\]

and

\[
\frac{\partial v}{\partial \theta} = -v_x r \sin \theta + v_y r \cos \theta.
\]

Assuming the Cauchy-Riemann equations in \((x, y)\) hold, we have \(u_x = v_y\) and \(u_y = -v_x\) at \((x_0, y_0)\). So from (5)

\[
v_r = v_x \cos \theta + v_y \sin \theta = -u_y \cos \theta + u_x \sin \theta \tag{5}
\]

and

\[
v_\theta = -v_x r \sin \theta + v_y r \cos \theta = u_y r \sin \theta + u_x r \cos \theta
\]

at \((r_0, \theta_0)\). Comparing (2) and (5) we have \(ru_r = v_\theta\) and \(u_\theta = -rv_r\) at \((r_0, \theta_0)\).
Lemma 2.23.A (continued)

Proof (continued). By the Chain Rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = u_x \cos \theta + u_y \sin \theta \quad \text{and} \quad (2)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = -u_x r \sin \theta + u_y r \cos \theta.$$  

Similarly,

$$\frac{\partial v}{\partial r} = v_x \cos \theta + v_y \sin \theta \quad \text{and} \quad \frac{\partial v}{\partial \theta} = -v_x r \sin \theta + v_y r \cos \theta. \quad (3)$$

Assuming the Cauchy-Riemann equations in \((x, y)\) hold, we have \(u_x = v_y\) and \(u_y = -v_x\) at \((x_0, y_0)\). So from (5)

$$v_r = v_x \cos \theta + v_y \sin \theta = -u_y \cos \theta + u_x \sin \theta \quad \text{and}$$

$$v_\theta = -v_x r \sin \theta + v_y r \cos \theta = u_y r \sin \theta + u_x r \cos \theta \quad (5)$$

at \((r_0, \theta_0)\). Comparing (2) and (5) we have \(ru_r = v_\theta\) and \(u_\theta = -rv_r\) at \((r_0, \theta_0)\).