Chapter 2. Analytic Functions
Section 2.24. Analytic Functions—Proofs of Theorems
Theorem 2.24.A.
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Theorem 2.24.A. If $f'(z) = 0$ everywhere in a domain $D$, then $f$ must be constant throughout $D$.

Proof. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Since $f'(z) = 0$ for all $z \in D$ (where $D$ is an open connected set), then $f$ is differentiable on $D$ and so satisfies the Cauchy-Riemann equations.
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Next, we consider $u(x, y)$ as a function of two real variables and approach it with some equipment from Calculus 3. Let $P$ be a point in $D$ and let $P'$ be another point in $D$ which lies on a line $L$ which lies in $D$. Let $\mathbf{U}$ denote the unit vector along line $L$ directed from $P$ to $P'$. Let $s$ denote the distance along $L$ from point $P$. See Figure 2.30.
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Next, we consider \( u(x, y) \) as a function of two real variables and approach it with some equipment from Calculus 3. Let \( P \) be a point in \( D \) and let \( P' \) be another point in \( D \) which lies on a line \( L \) which lies in \( D \). Let \( U \) denote the unit vector along line \( L \) directed from \( P \) to \( P' \). Let \( s \) denote the distance along \( L \) from point \( P \). See Figure 2.30.
Theorem 2.24.A (continued 1)

Proof (continued).

The directional derivative of $u(x, y)$ along line $L$ is then $\frac{du}{ds} = \text{grad}(u) \cdot \mathbf{U}$ where $\text{grad}(u) = \nabla u = u_x(x, y)\mathbf{i} + u_y(x, y)\mathbf{j}$ (see Theorem 9 in my Calculus 3 (MATH 2110) notes on 14.5. Directional Derivatives and Gradient Vectors).
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where \( \nabla u = u_x(x, y) \mathbf{i} + u_y(x, y) \mathbf{j} \) (see Theorem 9 in my Calculus 3 (MATH 2110) notes on 14.5. Directional Derivatives and Gradient Vectors). Since \( u_x(x, y) = u_y(x, y) = 0 \) for all \( (x, y) \in D \), then \( \nabla u = 0 \) at all points along \( L \). So \( u \) is constant on \( L \) and the value of \( u \) at point \( P \) is the same as its value at \( P' \).
Theorem 2.24.A (continued 1)

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The directional derivative of \( u(x, y) \) along line \( L \) is then \( \frac{du}{ds} = \text{grad}(u) \cdot \mathbf{U} \) where \( \text{grad}(u) = \nabla u = u_x(x, y)i + u_y(x, y)j \) (see Theorem 9 in my Calculus 3 (MATH 2110) notes on 14.5. Directional Derivatives and Gradient Vectors). Since \( u_x(x, y) = u_y(x, y) = 0 \) for all \( (x, y) \in D \), then \( \text{grad}(u) = 0 \) at all points along \( L \). So \( u \) is constant on \( L \) and the value of \( u \) at point \( P \) is the same as its value at \( P' \).
Theorem 2.24.A (continued 2)

**Theorem 2.24.A.** If $f'(z) = 0$ everywhere in a domain $D$, then $f$ must be constant throughout $D$.

**Proof (continued).** Since $D$ is an open connected set, then any two points in $D$ can be joined by a sequence of line segments in $D$ (the is Theorem II.2.3 in Conway’s *Functions of One Complex Variable I*; see my notes for Complex Analysis 1 on II.2. Connectedness). So if $P$ and $Q$ are any two points in $D$, then there is a sequence of line segments in $D$, say $PP_1, P_1P_2, \ldots, P_nQ$, joining $P$ to $Q$. As argued above, the value of $u$ is the same at each of the points $P, P_1, P_2, \ldots, P_n, Q$ and so the value of $u$ is the same at $P$ and $Q$. 

\[ u(x, y) = a \text{ for all } (x, y) \in D. \]

\[ v(x, y) = b \text{ for all } (x, y) \in D. \]

Therefore $f$ is constant on $D$ and $f(z) = a + ib$ for some $a + ib \in \mathbb{C}$. 

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Theorem 2.24.A (continued 2)

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**Proof (continued).** Since \( D \) is an open connected set, then any two points in \( D \) can be joined by a sequence of line segments in \( D \) (this is Theorem II.2.3 in Conway’s *Functions of One Complex Variable I*; see my notes for Complex Analysis 1 on II.2. Connectedness). So if \( P \) and \( Q \) are any two points in \( D \), then there is a sequence of line segments in \( D \), say \( PP_1, P_1P_2, \ldots, P_nQ \), joining \( P \) to \( Q \). As argued above, the value of \( u \) is the same at each of the points \( P, P_1, P_2, \ldots, P_n, Q \) and so the value of \( u \) is the same at \( P \) and \( Q \). Since \( P \) and \( Q \) are arbitrary points in \( D \), then \( u \) is constant on \( D \), say \( u(x, y) = a \) for all \( (x, y) \in D \).
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Similarly, since \( v_x(x, y) = v_y(x, y) = 0 \) on \( D \), then \( v(x, y) \) is constant on \( D \), say \( v(x, y) = b \) for all \( (x, y) \in D \). Therefore \( f \) is constant on \( D \) and \( f(z) = a + ib \) for some \( a + ib \in \mathbb{C} \).
Theorem 2.24.A (continued 2)

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Similarly, since \( v_x(x, y) = v_y(x, y) = 0 \) on \( D \), then \( v(x, y) \) is constant on \( D \), say \( v(x, y) = b \) for all \( (x, y) \in D \). Therefore \( f \) is constant on \( D \) and \( f(z) = a + ib \) for some \( a + ib \in \mathbb{C} \). □