Theorem 2.25.A

Suppose that a function \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) and its conjugate \( \overline{f(z)} = u(x, y) - iv(x, y) \) are both analytic in a given domain \( D \). Then \( f \) is constant throughout \( D \).

**Proof.** Write \( \overline{f(z)} = U(x, y) + iV(x, y) \), so that
\[
U(x, y) = u(x, y) \quad \text{and} \quad V(x, y) = -v(x, y).
\]  
Since \( f \) is analytic in \( D \) then the Cauchy-Riemann equations are satisfied, by Theorem 2.21.A, and so
\[
u_x(x, y) = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -v_x(x, y), \tag{2}
\]
for all \((x, y)\) in \( D \). Since \( \overline{f} \) is analytic in \( D \), then the Cauchy-Riemann equations also give
\[
u_x(x, y) = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -v_x(x, y) \quad \text{for all} \quad (x, y) \quad \text{in} \quad D.
\]
By equations (1), we therefore have throughout \( D \) that
\[
u_x(x, y) = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -(v_x(x, y)) \tag{4}
\]

Theorem 2.25.A (continued)

**Theorem 2.25.A.**

Suppose that a function \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) and its conjugate \( \overline{f(z)} = u(x, y) - iv(x, y) \) are both analytic in a given domain \( D \). Then \( f \) is constant throughout \( D \).

**Proof (continued).** Throughout \( D \) we have:
\[
u_x(x, y) = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -v_x(x, y) \tag{2}
\]
\[
u_x(x, y) = -v_y(x, y) \quad \text{and} \quad u_y(x, y) = v_x(x, y) \tag{4}
\]
Adding the respective sides of the first equations in (2) and (4) yields \( u_x = 0 \) on \( D \). Subtracting the respective sides of the second equations in (2) and (4) yields \( v_x = 0 \) on \( D \). So by Theorem 2.21.A, \( f'(z) = u_x(x, y) + iv_x(x, y) = 0 \) and so by Theorem 2.24.A, \( f \) is constant on \( D \).