Chapter 2. Analytic Functions
Section 2.26. Harmonic Functions—Proofs of Theorems
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Theorem 2.26.1

**Theorem 2.26.1.** If a function \( f(z) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \), then its component functions \( u(x, y) \) and \( v(x, y) \) are harmonic in \( D \).

**Proof.** In Corollary 4.52.A, we will see that if \( f(z) = u(x, y) + iv(x, y) \) is analytic at a point then \( u(x, y) \) and \( v(x, y) \) have continuous partial derivatives of all orders at the point. Since \( f \) is analytic in \( D \), then the Cauchy-Riemann equations are satisfied by Theorem 2.21.A so \( u_x = v_y \) and \( u_y = -v_x \).
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\[
    u_{xx} = v_{yx} \quad \text{and} \quad u_{yx} = -v_{xx}. \tag{3}
\]

Differentiating the Cauchy-Riemann equations with respect to \( y \) gives

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Proof (continued).

\[ u_{xx} = v_{yx} \text{ and } u_{yx} = -v_{xx}. \quad (3) \]
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By “The Mixed Derivative Theorem (Clairaut’s Theorem)” (see Theorem 2 of my Calculus 3 [MATH 2110] notes: http://faculty.etsu.edu/gardnerr/2110/notes-12e/c14s3.pdf) if the first partials and the mixed second partials are continuous then the mixed second partials are equal. So \( u_{xy} = u_{yx} \) and \( v_{yx} = v_{xy} \).
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\[ u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0. \]

So \( u(x, y) \) and \( v(x, y) \) are harmonic in \( D \).
Proof (continued).

\[ u_{xx} = v_{yx} \text{ and } u_{yx} = -v_{xx}. \quad (3) \]

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Theorem 2.26.2

**Theorem 2.26.2.** A function \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \) if and only if \( v(x, y) \) is a harmonic conjugate of \( u(x, y) \).

**Proof.** If \( v \) is a harmonic conjugate of \( u \), then their first order partial derivatives satisfy the Cauchy-Riemann equations (by definition of harmonic conjugates) throughout \( D \). So by Theorem 2.22.A, \( f \) is differentiable throughout \( D \) and so \( f \) is analytic on \( D \).
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If \( f \) is analytic in \( D \), then by Theorem 2.26.1 \( u \) and \( v \) are harmonic in \( D \). By the definition of analytic, \( f \) is differentiable throughout \( D \) and so by Theorem 2.21.A, \( u \) and \( v \) satisfy the Cauchy-Riemann equations on \( D \). So (by the definition of harmonic conjugates), \( v \) is a harmonic conjugate of \( u \).
Theorem 2.26.2. A function \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \) if and only if \( v(x, y) \) is a harmonic conjugate of \( u(x, y) \).

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