Complex Variables

Chapter 2. Analytic Functions
Section 2.26. Harmonic Functions—Proofs of Theorems
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Theorem 2.26.1. If a function \( f(z) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \), then its component functions \( u(x, y) \) and \( v(x, y) \) are harmonic in \( D \).

Proof. In Corollary 4.52.A, we will see that if \( f(z) = u(x, y) + iv(x, y) \) is analytic at a point then \( u(x, y) \) and \( v(x, y) \) have continuous partial derivatives of all orders at the point. Since \( f \) is analytic in \( D \) then by the definition of “analytic” \( f \) is differentiable on \( D \) and so the Cauchy-Riemann equations are satisfied by Theorem 2.21.A. So \( u_x = v_y \) and \( u_y = -v_x \) on \( D \).
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\[
 u_{xx} = v_{yx} \quad \text{and} \quad u_{yx} = -v_{xx}. \tag{3}
\]

Differentiating the Cauchy-Riemann equations with respect to \( y \) gives

\[
 u_{xy} = v_{yy} \quad \text{and} \quad u_{yy} = -v_{xy}. \tag{4}
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Theorem 2.26.1

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Proof. In Corollary 4.52.A, we will see that if $f(z) = u(x, y) + iv(x, y)$ is analytic at a point then $u(x, y)$ and $v(x, y)$ have continuous partial derivatives of all orders at the point. Since $f$ is analytic in $D$ then by the definition of “analytic” $f$ is differentiable on $D$ and so the Cauchy-Riemann equations are satisfied by Theorem 2.21.A. So $u_x = v_y$ and $u_y = -v_x$ on $D$. Differentiating the Cauchy-Riemann equations with respect to $x$ gives

$$u_{xx} = v_{yx} \text{ and } u_{yx} = -v_{xx}.$$ (3)

Differentiating the Cauchy-Riemann equations with respect to $y$ gives

$$u_{xy} = v_{yy} \text{ and } u_{yy} = -v_{xy}.$$ (4)
Theorem 2.26.1 (continued)

Proof (continued).

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By “The Mixed Derivative Theorem (Clairaut’s Theorem)” (see Theorem 2 of my Calculus 3 [MATH 2110] notes on 14.3. Partial Derivatives) if the first partials and the mixed second partials are continuous then the mixed second partials are equal. So \( u_{xy} = u_{yx} \) and \( v_{yx} = v_{xy} \).
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\[ u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0. \]

So \( u(x, y) \) and \( v(x, y) \) are harmonic in \( D \).\]
Theorem 2.26.1 (continued)

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\[ u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0. \]

So \( u(x, y) \) and \( v(x, y) \) are harmonic in \( D \). \( \square \)
Theorem 2.26.2. A function \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \) if and only if \( v(x, y) \) is a harmonic conjugate of \( u(x, y) \).

Proof. If \( v \) is a harmonic conjugate of \( u \), then their first order partial derivatives satisfy the Cauchy-Riemann equations (by definition of harmonic conjugates) throughout \( D \). So by Theorem 2.22.A, \( f \) is differentiable throughout \( D \) and so \( f \) is analytic on \( D \).
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If \( f \) is analytic in \( D \), then by Theorem 2.26.1 \( u \) and \( v \) are harmonic in \( D \). By the definition of analytic, \( f \) is differentiable throughout \( D \) and so by Theorem 2.21.A, \( u \) and \( v \) satisfy the Cauchy-Riemann equations on \( D \). So (by the definition of harmonic conjugates), \( v \) is a harmonic conjugate of \( v \).
Theorem 2.26.2. A function \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \) if and only if \( v(x, y) \) is a harmonic conjugate of \( u(x, y) \).

Proof. If \( v \) is a harmonic conjugate of \( u \), then their first order partial derivatives satisfy the Cauchy-Riemann equations (by definition of harmonic conjugates) throughout \( D \). So by Theorem 2.22.A, \( f \) is differentiable throughout \( D \) and so \( f \) is analytic on \( D \).

If \( f \) is analytic in \( D \), then by Theorem 2.26.1 \( u \) and \( v \) are harmonic in \( D \). By the definition of analytic, \( f \) is differentiable throughout \( D \) and so by Theorem 2.21.A, \( u \) and \( v \) satisfy the Cauchy-Riemann equations on \( D \). So (by the definition of harmonic conjugates), \( v \) is a harmonic conjugate of \( v \).