Chapter 3. Elementary Functions
Section 3.32. Some Identities Involving Logarithms—Proofs of Theorems
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Lemma 3.32.A. For the multiple-valued “function” \( \log z \) defined in Section 3.30, we have for all nonzero \( z_1, z_2 \in \mathbb{C} \) that

\[
\log(z_1 z_2) = \log z_1 + \log z_2.
\]

Proof. Since we have by definition, \( \log z = \ln |z| + \arg(z) \), and by Lemma 1.8.1, \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \), then

\[
\log(z_1 z_2) = \ln |z_1 z_2| + \arg(z_1 z_2) = \ln |z_1| + \ln |z_2| + \arg(z_1) + \arg(z_2)
\]

\[
= (\ln |z_1| + \arg(z_1)) + (\ln |z_2| + \arg(z_2)) = \log z_1 + \log z_2.
\]
Lemma 3.32.A. For the multiple-valued “function” log \( z \) defined in Section 3.30, we have for all nonzero \( z_1, z_2 \in \mathbb{C} \) that

\[
\log(z_1 z_2) = \log z_1 + \log z_2.
\]

Proof. Since we have by definition, \( \log z = \ln |z| + \arg(z) \), and by Lemma 1.8.1, \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \), then

\[
\log(z_1 z_2) = \ln |z_1 z_2| + \arg(z_1 z_2) = \ln |z_1| + \ln |z_2| + \arg(z_1) + \arg(z_2)
\]

\[
= (\ln |z_1| + \arg(z_1)) + (\ln |z_2| + \arg(z_2)) = \log z_1 + \log z_2.
\]

\[\square\]
Lemma 3.32.C

Lemma 3.32.C. For any nonzero $z \in \mathbb{C}$, for all $n \in \mathbb{Z}$ we have $z^n = e^{n \log z}$.

Proof. First, for $z = re^{i\theta} = |z|e^{i \arg z}$, notice that $e^{\log z} = e^{\ln |z| + i \arg z} = e^{\ln |z|}e^{i \arg z} = |z|e^{i \arg z} = z$. So for $n \geq 0$ we have

$$z^n = (e^{\log z})^n = \underbrace{(e^{\log z})(e^{\log z}) \ldots (e^{\log z})}_{n \text{ times}} = e^{n \log z} \text{ by Lemma 3.29.A.}$$

For $n < 0$ we have

$$z^n = (e^{\log z})^n = e^{-n \log z} \text{ by Lemma 3.29.A.}$$
Lemma 3.32.C

Lemma 3.32.C. For any nonzero \( z \in \mathbb{C} \), for all \( n \in \mathbb{Z} \) we have
\[
z^n = e^{n \log z}.
\]

Proof. First, for \( z = re^{i\theta} = |z|e^{i \arg z} \), notice that
\[
e^{\log z} = e^{\ln |z| + i \arg z} = e^{\ln |z|}e^{i \arg z} = |z|e^{i \arg z} = z.
\]
So for \( n \geq 0 \) we have
\[
z^n = (e^{\log z})^n = \left( e^{\log z} \right) \left( e^{\log z} \right) \cdots \left( e^{\log z} \right)
\]
\[
= e^{n \log z} \text{ by Lemma 3.29.A.}
\]

For \( n < 0 \) we have
\[
z^n = (e^{\log z})^n = (e^{-\log z})^{-n} = \left( e^{-\log z} \right) \left( e^{-\log z} \right) \cdots \left( e^{-\log z} \right)
\]
\[
= e^{-(-n) \log z} \text{ by Lemma 3.29.A}
\]
\[
= e^{n \log z}.
\]
Lemma 3.32.C

**Lemma 3.32.C.** For any nonzero \( z \in \mathbb{C} \), for all \( n \in \mathbb{Z} \) we have \( z^n = e^{n \log z} \).

**Proof.** First, for \( z = re^{i\theta} = |z|e^{i\arg z} \), notice that
\[
e^{\log z} = e^{\ln |z| + i\arg z} = e^{\ln |z|}e^{i\arg z} = |z|e^{i\arg z} = z.
\]
So for \( n \geq 0 \) we have
\[
z^n = (e^{\log z})^n = (e^{\log z})(e^{\log z}) \cdots (e^{\log z}) \quad \text{\( n \) times}
\]
\[= e^{n \log z} \text{ by Lemma 3.29.A.}\]

For \( n < 0 \) we have
\[
z^n = (e^{\log z})^n = (e^{-\log z})^{-n} = (e^{-\log z})(e^{-\log z}) \cdots (e^{-\log z}) \quad \text{\( -n \) times}
\]
\[= e^{-(\log z)^{-n}} \text{ by Lemma 3.29.A}
\]
\[= e^{n \log z}.\]
Lemma 3.32.D

Lemma 3.32.D. For any nonzero $z \in \mathbb{C}$, we have that for $n = 1, 2, 3, \ldots$

$$\exp\left(\frac{1}{n} \log z\right)$$

is a set consisting of $n$ distinct elements each of which is an $n$th root of $z$ (that is, when raised to the $n$th power gives $z$).

Proof. Let $z = r \exp(i\Theta) = |z| \exp(i\Theta)$ where $\Theta$ is the principal value of $\arg(z)$ (that is, $\Theta \in \arg(z)$ and $-\pi < \Theta \leq \pi$). Then

$$\exp\left(\frac{1}{n} \log z\right) = \exp\left(\frac{1}{n} (\ln |z| + i(\Theta + 2k\pi))\right)\text{ where } k \in \mathbb{Z}$$

$$= \exp\left(\frac{1}{n} \ln |z| + i \frac{\Theta + 2k\pi}{n}\right)$$

$$= \exp\left(\frac{1}{n} \ln |z|\right) \exp\left(i \frac{\Theta + 2k\pi}{n}\right)\text{ by Lemma 3.29.A. (7)}$$
Lemma 3.32.D. For any nonzero \( z \in \mathbb{C} \), we have that for \( n = 1, 2, 3, \ldots \)

\[
\exp\left(\frac{1}{n} \log z\right)
\]

is a set consisting of \( n \) distinct elements each of which is an \( n \)th root of \( z \) (that is, when raised to the \( n \)th power gives \( z \)).

**Proof.** Let \( z = r \exp(i\Theta) = |z| \exp(i\Theta) \) where \( \Theta \) is the principal value of \( \arg(z) \) (that is, \( \Theta \in \arg(z) \) and \( -\pi < \Theta \leq \pi \)). Then

\[
\exp\left(\frac{1}{n} \log z\right) = \exp\left(\frac{1}{n} (\ln |z| + i(\Theta + 2k\pi))\right) \text{ where } k \in \mathbb{Z}
\]

\[
= \exp\left(\frac{1}{n} \ln |z| + i \frac{\Theta + 2k\pi}{n}\right)
\]

\[
= \exp\left(\frac{1}{n} \ln |z|\right) \exp\left(i \frac{\Theta + 2k\pi}{n}\right) \text{ by Lemma 3.29.A. (7)}
\]
Lemma 3.32.D (continued)

Proof (continued). Now
\[ \exp \left( i \left( \Theta/n + 2k\pi/n \right) \right) = \cos \left( \Theta/n + 2k\pi/n \right) + i \sin \left( \Theta/n + 2k\pi/n \right) \]
and this results in \( n \) distinct values as \( k \) ranges over the distinct values modulo \( n \) (say, \( k = 0, 1, \ldots, n - 1 \)). For each value given in (7), we have

\[
\left[ \exp \left( \frac{1}{n} \log z \right) \right]^n = \left[ \exp \left( \frac{1}{n} \ln |z| \right) \exp \left( i \frac{\Theta + 2k\pi}{n} \right) \right]^n = \left[ \exp \left( \frac{1}{n} \ln |z| \right) \right]^n \left[ \exp \left( i \frac{\Theta + 2k\pi}{n} \right) \right]^n
\]

\[ = \exp(\ln |z|) \exp(i(\Theta + 2k\pi)) \text{ by Lemma 3.29.A} \]
\[ = |z|e^{i(\Theta + 2k\pi)} = |z|e^{i\Theta} = z. \]

So the result follows, which we denote as \( z^{1/n} = \exp \left( \frac{1}{n} \log z \right) \). \( \Box \)
Lemma 3.32.D (continued)

Proof (continued). Now
\[
\exp \left( i \left( \Theta / n + 2k \pi / n \right) \right) = \cos(\Theta / n + 2k \pi / n) + i \sin(\Theta / n + 2k \pi / n)
\]
and this results in \( n \) distinct values as \( k \) ranges over the distinct values modulo \( n \) (say, \( k = 0, 1, \ldots, n - 1 \)). For each value given in (7), we have

\[
\left[ \exp \left( \frac{1}{n} \log z \right) \right]^n = \left[ \exp \left( \frac{1}{n} \ln |z| \right) \exp \left( i \frac{\Theta + 2k \pi}{n} \right) \right]^n
\]
\[
= \left[ \exp \left( \frac{1}{n} \ln |z| \right) \right]^n \left[ \exp \left( i \frac{\Theta + 2k \pi}{n} \right) \right]^n
\]
\[
= \exp(\ln |z|) \exp(i(\Theta + 2k \pi)) \text{ by Lemma 3.29.A}
\]
\[
= |z|e^{i(\Theta+2k\pi)} = |z|e^{i\Theta} = z.
\]

So the result follows, which we denote as \( z^{1/n} = \exp \left( \frac{1}{n} \log z \right) \).