Theorem 33.3.A. For any branch of $z^c$, we have \( \frac{d}{dz}[z^c] = cz^{c-1} \) where the branch of $z^{c-1}$ is based on the same branch of the logarithm on which $z^c$ is based.

Proof. Let $\log z$ represent some branch of the logarithm so that $\log z = \ln|z| + i\theta$ where $\theta \in \text{arg}(z)$ and $\alpha < \theta < \alpha + 2\pi$ for some given $\alpha$. Then

\[
\frac{d}{dz}[z^c] = \frac{d}{dz}[\exp(c \log z)] \\
= c \frac{1}{z}[\exp(c \log z)] \text{ since } \frac{d}{dz}[\log z] = 1/z \text{ as shown in Note 3.31.A} \\
= \frac{1}{z} \exp(c \log z) \text{ and by the Chain Rule (Theorem 2.20.C} \\
= c \exp(c \log z) \text{ since } z = \exp(\log z) \text{ as shown in Note 3.30.A} \\
= \ldots
\]

Theorem 33.3.A (continued)

\[
\frac{d}{dz}[z^c] = c \exp(c \log z - \log z) = c \exp((c - 1) \log z) \text{ by Lemma 3.32.B} \\
= cz^{c-1} \text{ for the branch of } z^{c-1} \text{ based on the branch of the logarithm } \log z.
\]