Chapter 3. Elementary Functions
Section 3.33. Complex Exponents—Proofs of Theorems
Theorem 3.33.A

Theorem 3.33.A. For any branch of $z^c$, we have $\frac{d}{dz}[z^c] = cz^{c-1}$ where the branch of $z^{c-1}$ is based on the same branch of the logarithm on which $z^c$ is based.

Proof. Let $\log z$ represent some branch of the logarithm so that $\log z = \ln |z| + i\theta$ where $\theta \in \arg(z)$ and $\alpha < \theta < \alpha + 2\pi$ for some given $\alpha$. Then

\[
\frac{d}{dz}[z^c] = \frac{d}{dz}[\exp(c \log z)]
\]

\[
= c \frac{1}{z} [\exp(c \log z)] \quad \text{since} \quad \frac{d}{dz}[\log z] = 1/z \quad \text{as shown in Note 3.31.A}
\]

and by the Chain Rule (Theorem 2.20.C

\[
= c \frac{\exp(c \log z)}{\exp(\log z)} \quad \text{since} \quad z = \exp(\log z) \quad \text{as shown in Note 3.30.A}
\]

\[
= \ldots
\]
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**Proof.** Let $\log z$ represent some branch of the logarithm so that $\log z = \ln |z| + i\theta$ where $\theta \in \arg(z)$ and $\alpha < \theta < \alpha + 2\pi$ for some given $\alpha$. Then

$$\frac{d}{dz}[z^c] = \frac{d}{dz}[\exp(c \log z)]$$

$$= c \frac{1}{z} [\exp(c \log z)] \text{ since } \frac{d}{dz} [\log z] = 1/z \text{ as shown in Note 3.31.A}$$

and by the Chain Rule (Theorem 2.20.C)

$$= c \frac{\exp(c \log z)}{\exp(\log z)} \text{ since } z = \exp(\log z) \text{ as shown in Note 3.30.A}$$

$$= \ldots$$
Theorem 3.33.A. For any branch of $z^c$, we have $\frac{d}{dz}[z^c] = cz^{c-1}$ where the branch of $z^{c-1}$ is based on the same branch of the logarithm on which $z^c$ is based.

Proof (continued). $\ldots$

$$\frac{d}{dz}[z^c] = c \exp(c \log z - \log z) = c \exp((c - 1) \log z)$$ by Lemma 3.32.B

$$= cz^{c-1}$$ for the branch of $z^{c-1}$ based on the branch of the logarithm $\log z$. 

$\square$