Chapter 4. Integrals
Section 4.48. Simply Connected Domains—Proofs of Theorems
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**Theorem 4.48.A.** If a function $f$ is analytic throughout a simply connected domain $D$, then $\int_C f(z) \, dz = 0$ for every closed contour $C$ lying in $D$.

**Proof for Some Closed Contours.** If $C$ is a simple closed contour then the claim holds by the Cauchy-Goursat Theorem (Theorem 4.46.A) since the points interior to $C$ are all in $D$. 
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Corollary 4.48.B

Corollary 4.48.B. A function $f$ that is analytic throughout a simply connected domain $D$ must have an antiderivative everywhere in $D$.

Proof. Let $C$ be a closed contour in $D$. Then by Theorem 4.48.A, $\int_C f(z) \, dz = 0$. Since $f$ is analytic throughout $D$ then $f$ is continuous on $D$. So by Theorem 4.44.A (the (c) implies (a) part), $f$ has an antiderivative throughout $D$. \qed
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