Complex Variables

Chapter 4. Integrals
Section 4.50. Cauchy Integral Formula—Proofs of Theorems

|z_1 + z_2| \leq |z_1| + |z_2|
Theorem 4.50.A. Cauchy Integral Formula
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Let $f$ be analytic everywhere inside and on simple closed contour $C$, parameterized in the positive sense. If $z_0$ is any point interior to $C$, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z) \, dz}{z - z_0}.$$ 

**Proof.** Let $C_\rho$ denote the positively oriented circle $|z - z_0| = \rho$, where $\rho$ is small enough the $C_\rho$ is interior to $C$ (which can be done since $C$ is a closed set and so $\mathbb{C} \setminus C$ is open with $z_0$ as an interior point of the open set $\mathbb{C} \setminus C$; see Figure 66).
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Proof. Let $C_\rho$ denote the positively oriented circle $|z - z_0| = \rho$, where $\rho$ is small enough the $C_\rho$ is interior to $C$ (which can be done since $C$ is a closed set and so $\mathbb{C} \setminus C$ is open with $z_0$ as an interior point of the open set $\mathbb{C} \setminus C$; see Figure 66).
Theorem 4.50.A (continued 1)

Proof (continued). The function \( f(z)/(z - z_0) \) is analytic inside and on \( C \) except at \( z_0 \). So by the Principle of Deformation (Corollary 4.49.B),
\[
\int_C \frac{f(z)}{z - z_0} \, dz = \int_{C_\rho} \frac{f(z)}{z - z_0} \, dz.
\]
So
\[
\int_C \frac{f(z)}{z - z_0} \, dz - f(z_0) \int_{C_\rho} \frac{dz}{z - z_0} = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} \, dz.
\]
Next,
\[
\int_{C_\rho} \frac{dz}{z - z_0} = 2\pi i \text{ by Exercise 42.10(b), so}
\]
\[
\int_C \frac{f(z)}{z - z_0} \, dz - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} \, dz. \quad (4)
\]
Since \( f \) is analytic, then it is continuous at \( z_0 \) and so for all \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( |z - z_0| < \delta \) then \( |f(z) - f(z_0)| < \varepsilon/(2\pi) \).
Proof (continued). The function \( f(z)/(z - z_0) \) is analytic inside and on \( C \) except at \( z_0 \). So by the Principle of Deformation (Corollary 4.49.B),
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So
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\int_C \frac{f(z) \, dz}{z - z_0} - f(z_0) \int_{C_\rho} \frac{dz}{z - z_0} = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} \, dz.
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Since \( f \) is analytic, then it is continuous at \( z_0 \) and so for all \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( |z - z_0| < \delta \) then \( |f(z) - f(z_0)| < \varepsilon/(2\pi) \). The only restriction on \( \rho \) above is that \( C_\rho \) is interior to \( C \). Let \( \rho' = \min\{\rho, \delta/2\} \). Then \( C_{\rho'} \) is interior to \( C \) and so the equations above involving \( C_\rho \) also hold for \( C_{\rho'} \).
Theorem 4.50.A (continued 1)

**Proof (continued).** The function \( f(z)/(z - z_0) \) is analytic inside and on \( C \) except at \( z_0 \). So by the Principle of Deformation (Corollary 4.49.B),

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\int_C \frac{f(z)}{z - z_0} \, dz = \int_{C_\rho} \frac{f(z)}{z - z_0} \, dz.
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So

\[
\int_C \frac{f(z)}{z - z_0} \, dz - f(z_0) \int_{C_\rho} \frac{dz}{z - z_0} = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} \, dz.
\]

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\int_{C_\rho} \frac{dz}{z - z_0} = 2\pi i \text{ by Exercise 42.10(b), so}
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\]  \hspace{1cm} (4)

Since \( f \) is analytic, then it is continuous at \( z_0 \) and so for all \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( |z - z_0| < \delta \) then \( |f(z) - f(z_0)| < \varepsilon/(2\pi) \). The only restriction on \( \rho \) above is that \( C_\rho \) is interior to \( C \). Let \( \rho' = \min\{\rho, \delta/2\} \). Then \( C_{\rho'} \) is interior to \( C \) and so the equations above involving \( C_\rho \) also hold for \( C_{\rho'} \).
**Theorem 4.50.A (continued 2)**

**Proof (continued).** Then for \( z \) on \( C_{\rho'} \) we have \( |z - z_0| = \rho' \leq \delta/2 < \delta \) and so \( |f(z) - f(z_0)| < \varepsilon/(2\pi) \); also the length of \( C_{\rho'} \) is \( 2\pi \rho' \) and so by Theorem 4.43.A,

\[
\left| \int_{C_{\rho'}} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| \leq \left( \frac{2/(2\pi)}{\rho'} \right) (2\pi \rho') = \varepsilon.
\]

So by equation (4),

\[
\left| \int_{C_{\rho'}} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| = \left| \int_C \frac{f(z) \, dz}{z - z_0} - 2\pi i f(z_0) \right| < \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, then the quantity \( \int_C \frac{f(z) \, dz}{z - z_0} - 2\pi i f(z_0) \) must be 0, and the result follows.
Theorem 4.50.A (continued 2)

Proof (continued). Then for $z$ on $C_{\rho'}$ we have $|z - z_0| = \rho' \leq \delta/2 < \delta$ and so $|f(z) - f(z_0)| < \varepsilon/(2\pi)$; also the length of $C_{\rho'}$ is $2\pi \rho'$ and so by Theorem 4.43.A,

$$\left| \int_{C_{\rho'}} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| \leq \left( \frac{2/(2\pi)}{\rho'} \right) (2\pi \rho') = \varepsilon.$$ 

So by equation (4),

$$\left| \int_{C_{\rho'}} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| = \left| \int_{C} \frac{f(z) \, dz}{z - z_0} - 2\pi i f(z_0) \right| < \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, then the quantity $\int_{C} \frac{f(z) \, dz}{z - z_0} - 2\pi i f(z_0)$ must be 0, and the result follows.