Complex Variables

Chapter 4. Integrals
Section 4.51. An Extension of the Cauchy Integral Formula—Proofs of Theorems

Lemma 4.51.A

**Lemma 4.51.A.** Let \( f \) be analytic inside and on a simple closed contour \( C \), taken in the positive sense. If \( z \) is any point interior to \( C \) then \( f'(z) \) exists and
\[
f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2} \, ds.
\]

**Proof.** Let \( d \) be the smallest distance from \( z \) to points \( s \) on \( C \) and assume \( 0 < |\Delta z| < d \) (see Figure 67); the minimum distance \( d \) exists because \( C \) is a “compact set.”

![Figure 67](image)

**Proof (continued).** By the Cauchy Integral Formula (Theorem 4.50.A),
\[
f(z) = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{s-z},
\]
so
\[
\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{\Delta z} \left( \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{s - (z + \Delta z)} - \int_C \frac{f(s) \, ds}{s-z} \right)
\]
\[
= \frac{1}{2\pi i} \int_C \left( \frac{1}{s-z-\Delta z} - \frac{1}{s-z} \right) f(s) \, ds = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s-z-\Delta z)(s-z)}.
\]
Now
\[
\frac{1}{(s-z-\Delta z)(s-z)} = \frac{1}{(s-z)^2} + \frac{\Delta z}{(s-z-\Delta z)(s-z)^2},
\]
so
\[
\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s-z)^2} + \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s-z-\Delta z)(s-z)^2}.
\]

![Figure 68](image)

**Lemma 4.51.A (continued 2)**

**Proof (continued).**
\[
= \frac{1}{2\pi i} \int_C \left( \frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2} \right) f(s) \, ds
\]
\[
= \frac{1}{2\pi i} \int_C \frac{\Delta zf(s) \, ds}{(s-z-\Delta z)(s-z)^2}, \quad (*)
\]

Next, let \( M \) denote the maximum value of \(|f(s)|\) on \( C \) (which exists since \(|f(s)|\) is continuous and \( C \) is compact) and observe that since \(|s-z| > d\) (by the choice of \( d \) as a minimum distance) and \(|\Delta z| < d\) (by the choice of \( \Delta z \)) then
\[
|s-z-\Delta z| = |(s-z) - \Delta z| \geq |s-z| - |\Delta z| \geq d - |\Delta z| > 0.
\]
Lemma 4.51.A (continued 3)

**Proof (continued).** Thus by Theorem 4.43.A

\[ \left| \int_C \frac{\Delta z f(s) \, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{(d - |\Delta z|) d^2} L \]

where \( L \) is the length of \( C \). So from (*), this implies

\[ \left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s - z)^2} \right| \leq \frac{1}{2\pi} \left| \int_C \frac{\Delta z f(s) \, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{2\pi (d - |\Delta z|) d^2} L \]

and so as \( \Delta z \to 0 \) we see that \( \frac{|\Delta z| M}{2\pi (d - |\Delta z|) d^2} L \to 0 \). Hence,

\[ f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2} \, ds. \]

Therefore, \( f'(z) \) exists and has the claimed value. \( \square \)