Chapter 4. Integrals
Section 4.51. An Extension of the Cauchy Integral Formula—Proofs of Theorems
Table of contents

1 Lemma 4.51.A
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**Lemma 4.51.A.** Let $f$ be analytic inside and on a simple closed contour $C$, taken in the positive sense. If $z$ is any point interior to $C$ then $f'(z)$ exists and

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2} \, ds.$$  

**Proof.** Let $d$ be the smallest distance from $z$ to points $s$ on $C$ and assume $0 < |\Delta z| < d$ (see Figure 67); the minimum distance $d$ exists because $C$ is a “compact set.”
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![Diagram](image.png)
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Lemma 4.51.A (continued 1)

**Proof (continued).** By the Cauchy Integral Formula (Theorem 4.50.A),

\[ f(z) = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{s - z}, \]

so

\[ \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{\Delta z} \left( \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{s - (z + \Delta z)} - \int_C \frac{f(s) \, ds}{s - z} \right) \]

\[ = \frac{1}{2\pi i} \int_C \left( \frac{1}{s - z - \Delta z} - \frac{1}{s - z} \right) \frac{f(s)}{\Delta z} \, ds = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s - z - \Delta z)(s - z)}. \]

Now

\[ \frac{1}{(s - z - \Delta z)(s - z)} = \frac{1}{(s - z)^2} + \frac{\Delta z}{(s - z - \Delta z)(s - z)^2}, \]

so

\[ \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s - z)^2} \]

\[ = \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s - z - \Delta z)(s - z)} - \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s - z)^2}. \]
Lemma 4.51.A (continued 2)

Proof (continued).

\[= \frac{1}{2\pi i} \int_C \left( \frac{1}{(s - z - \Delta z)(s - z)} - \frac{1}{(s - z)^2} \right) f(s) \, ds\]

\[= \frac{1}{2\pi i} \int_C \frac{\Delta z f(s)}{(s - z - \Delta z)(s - z)^2} \quad (*)\]

Next, let \(M\) denote the maximum value of \(|f(s)|\) on \(C\) (which exists since \(|f(s)|\) is continuous and \(C\) is compact) and observe that since \(|s - z| > d\) (by the choice of \(d\) as a minimum distance) and \(|\Delta z| < d\) (by the choice of \(\Delta z\)) then

\[|s - z - \Delta z| = |(s - z) - \Delta z| \geq ||s - z| - |\Delta z|| \quad \text{by Corollary 1.4.1}\]

\[\geq |s - z| - |\Delta z| \geq d - |\Delta z| > 0.\]
Lemma 4.51.A (continued 2)

Proof (continued).

\[
= \frac{1}{2\pi i} \int_C \left( \frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2} \right) f(s) \, ds
\]

\[
= \frac{1}{2\pi i} \int_C \frac{\Delta z f(s) \, ds}{(s-z-\Delta z)(s-z)^2}. \quad (*)
\]

Next, let \( M \) denote the maximum value of \(|f(s)|\) on \( C \) (which exists since \(|f(s)|\) is continuous and \( C \) is compact) and observe that since \(|s-z| > d\) (by the choice of \( d \) as a minimum distance) and \(|\Delta z| < d\) (by the choice of \( \Delta z \)) then

\[
|s - z - \Delta z| = |(s - z) - \Delta z| \geq ||s - z| - |\Delta z|| \quad \text{by Corollary 1.4.1}
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\[
\geq |s - z| - |\Delta z| \geq d - |\Delta z| > 0.
\]
Lemma 4.51.A (continued 3)

Proof (continued). Thus by Theorem 4.43.A

\[ \left| \int_C \frac{\Delta z f(s) \, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{\left| \Delta z \right| M}{(d - \left| \Delta z \right|)d^2} L \]

where \( L \) is the length of \( C \). So from (\(*\)), this implies

\[ \left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s - z)^2} \right| \leq \frac{1}{2\pi} \left| \int_C \frac{\Delta z f(s) \, ds}{(s - z - \Delta z)(s - z)^2} \right| \]

\[ \leq \frac{\left| \Delta z \right| M}{2\pi(d - \left| \Delta z \right|)d^2} L \]

and so as \( \Delta z \to 0 \) we see that \( \frac{\left| \Delta z \right| M}{2\pi(d - \left| \Delta z \right|)d^2} L \to 0 \). Hence,

\[ f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2} \, ds. \]

Therefore, \( f'(z) \) exists and has the claimed value. \( \square \)
Lemma 4.51.A (continued 3)

Proof (continued). Thus by Theorem 4.43.A

\[ \left| \int_C \frac{\Delta zf(s) \, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{(d - |\Delta z|)d^2 L} \]

where \( L \) is the length of \( C \). So from (*) this implies

\[ \left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s) \, ds}{(s - z)^2} \right| = \frac{1}{2\pi} \left| \int_C \frac{\Delta zf(s) \, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{2\pi(d - |\Delta z|)d^2 L} \]

and so as \( \Delta z \to 0 \) we see that \( \frac{|\Delta z| M}{2\pi(d - |\Delta z|)d^2 L} \to 0 \). Hence,

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Therefore, \( f'(z) \) exists and has the claimed value. \( \square \)