Chapter 4. Integrals
Section 4.53. Liouville’s Theorem and the Fundamental Theorem of Algebra—Proofs of Theorems
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Theorem 4.53.1. Liouville’s Theorem.
If a function $f$ is entire and bounded in the whole complex plane, then $f$ is constant throughout the entire complex plane.

Proof. Let $f$ be a bounded entire function, say $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By Cauchy’s Inequality (Theorem 4.52.3) with $n = 1$, we have that for any $z_0 \in \mathbb{C}$ and, since $f$ is entire, for all $R > 0$, it must be that $|f'(z_0)| \leq M/R$. 
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Theorem 4.53.2. The Fundamental Theorem of Algebra.

Any complex polynomial \( P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \), where \( a_n \neq 0 \), of degree \( n \geq 1 \) has at least one zero. That is, there exists at least one point \( z_0 \in \mathbb{C} \) such that \( P(z_0) = 0 \).

**Proof.** ASSUME no such \( z_0 \) exists and that \( P(z) \) is nonzero throughout \( \mathbb{C} \). Then by Lemma 2.24.A, the function \( 1/P(z) \) is analytic throughout \( \mathbb{C} \); that is, \( 1/P(z) \) is an entire function.
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We claim that \( 1/P(z) \) is bounded. Notice that

\[
P(z) = \left( \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} + a_n \right) z^n.
\]

Since

\[
\lim_{z \to 0} \left( \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right) = 0,
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then for \( \varepsilon = |a_n|/2 \) there is \( R > 0 \) such that for all \( |z| > R \) we have...
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Theorem 4.53.2 (continued 1)

Proof (continued)... 

\[ \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right| < \frac{|a_n|}{2} = \varepsilon. \]

So for \(|z| > R|,

\[ \left| \left( \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right) + a_n \right| \]

\[ \geq \left| \left( \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right) - |a_n| \right| \quad \text{by Corollary 1.4.1} \]

\[ > |a_n|/2. \]

So

\[ |P(z)| = \left| \left( \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right) + a_n \right| |z^n| \]

\[ = \left| \left( \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right) + a_n \right| |z|^n \]

\[ > |a_n||z|^n/2 > |a_n|R^n/2 \quad \text{for} \quad |z| > R. \]
Theorem 4.53.2 (continued 1)

Proof (continued). ... 

\[ \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right| < \frac{|a_n|}{2} = \varepsilon. \]

So for \(|z| > R|\),

\[ \left\| \left( \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right) + a_n \right\| \]

\[ \geq \left\| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right\| - |a_n| \text{ by Corollary 1.4.1} \]

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So

\[ |P(z)| = \left\| \left\{ \left( \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right) + a_n \right\} z^n \right\| \]

\[ = \left\| \left( \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \right) + a_n \right\| |z|^n \]

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Proof (continued). So \( |1/P(z)| < 2/(|a_n|R^n) \) for \( |z| > R \). Now \( 1/P(z) \) is continuous by assumption and so by Theorem 2.18.3, \( |1/P(z)| \) is bounded, by say \( M \), on the closed and bounded set \( |z| \leq R \). Therefore

\[
\left| \frac{1}{P(z)} \right| \leq \begin{cases} 
|a_n|R^n/2 & \text{for } |z| > R \\
M & \text{for } |z| \leq R 
\end{cases}
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and \( 1/P(z) \) is a bounded entire function.
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$$\left|\frac{1}{P(z)}\right| \leq \begin{cases} |a_n|R^n/2 & \text{for } |z| > R \\ M & \text{for } |z| \leq R \end{cases}$$

and $1/P(z)$ is a bounded entire function.

But Liouville’s Theorem then implies that $1/P(z)$ is constant, a CONTRADICTION. So the assumption that $P(z)$ is nonzero throughout $\mathbb{C}$ is false and there must be some $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$. \qed
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