Lemma 4.54.A

Lemma 4.54.A. Suppose that \(|f(z)| \leq |f(z_0)|\) at each point \(z\) in some neighborhood \(|z - z_0| < \varepsilon\) in which \(f\) is analytic. Then \(|f(z)|\) has the constant value \(f(z_0)\) throughout that neighborhood.

Proof. Let \(z_1 \neq z_0\) be in the \(\varepsilon\)-neighborhood of \(z_0\). Let \(\rho = |z_1 - z_0|\). Let \(C_{\rho}\) be the positively oriented circle \(|z - z_0| = \rho\). Then \(f\) is analytic on and inside \(C_{\rho}\) and so by the Cauchy Integral Formula (Theorem 4.50.A),

\[
f(z_0) = \frac{1}{2\pi i} \int_{C_{\rho}} \frac{f(z)}{z - z_0} \, dz.
\]

Parameterize \(C_{\rho}\) as \(z = z_0 + \rho e^{i\theta}, \, \theta \in [0, 2\pi]\).

Then

\[
f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{(z_0 + \rho e^{i\theta}) - z_0} \, i\rho e^{i\theta} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \, d\theta.
\]

we then have from (2) that

\[
|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta \text{ by Lemma 4.43.A. (3)}
\]

Lemma 4.51.A (continued)

Proof (continued). On the other hand, by hypothesis,

\[
|f(z_0 + \rho e^{i\theta})| \leq |f(z_0)| \text{ for } \theta \in [0, 2\pi]
\]

so that

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| \, d\theta = 2\pi |f(z_0)|,
\]

or

\[
|f(z_0)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta.
\]

Combining equations (3) and (5) gives

\[
|f(z_0)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta,
\]

or

\[
|f(z_0)| - \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta = 0.
\]

Now the integrand is nonnegative by hypothesis and is a continuous function of \(\theta\) on \([0, 2\pi]\). If the integral of a continuous real-valued nonnegative function over some interval is 0 then the function must be identically 0 (yes, we could use a reference for this). So

\[
|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta,
\]

for all \(\theta \in [0, 2\pi]\). That is,

\[
|f(z_0)| = |f(z)|\]

for all \(z \in C_{\rho}\). In particular, \(|f(z_0)| = |f(z_1)|\). Since \(z_1\) is an arbitrary point in the \(\varepsilon\)-neighborhood of \(z_0\), then \(|f(z_0)| = |f(z)|\) for all \(z\) such that \(|z - z_0| < \varepsilon\). So by Example 2.25.4/Theorem 2.25.B,

\[
f(z) = f(z_0)\]

for all \(z\) satisfying \(|z - z_0| < \varepsilon\), as claimed. □
Theorem 4.54.C (continued)

Proof (continued).

Since $|f(z_0)|$ is a maximum of $|f(z)|$ on $D$, then it is a maximum on $N_0$ and $z_0 \in N_0$, so by Lemma 4.54.A, $f$ is constant on $N_0$. In particular, $f(z_1) = f(z_0)$. So $|f(z_1)|$ is a maximum of $|f(z)|$ on $N_1$ and $z_1 \in N_1$, so by Lemma 4.54.A, $f$ is a constant on $N_1$. Inductively, $f$ is constant on $N_0 \cup N_1 \cup \cdots \cup N_n$ and so $f(z_0) = f(P)$. Since $P$ is an arbitrary point of $D$, then $f$ is constant on $D$, as claimed.

\[ \square \]

Theorem 4.54.D

**Theorem 4.54.D. Maximum Modulus Theorem, Alternative Version.** Suppose that a function $f$ is continuous on a closed bounded region $R$ and that it is analytic and not constant in the interior of $R$. Then the maximum value of $|f(z)|$ on $R$, which is always reached (by Theorem 2.18.3) occurs somewhere on the boundary of $R$ and never in the interior.

**Proof.** Let $M$ be the maximum of $|f(z)|$ on $R$, so that $|f(z)| \leq M$ for all $z \in R$. If $f$ is constant, then $|f(z)| = M$ for all $z \in R$ and so the maximum is attained on the boundary. If $f$ is not constant, then by the Maximum Modulus Theorem (Theorem 4.54.C) the maximum of $|f(z)|$ cannot be attained for some $z_0$ in the interior of $R$. Since the maximum is attained somewhere on $R$ by Theorem 2.18.3, then it must be attained on the boundary of $R$ (recall that a “region” is an open connected set along with some, none, or all of its boundary points).

\[ \square \]

Theorem 4.54.E

**Theorem 4.54.E.** Let $f$ be continuous on a closed bounded region $R$, and analytic and not constant on the interior of $R$. For $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, function $u(x, y)$ attains its maximum value in $R$ on the boundary of $R$ and not in the interior.

**Proof.** Let $g(z) = e^{f(z)}$. Then $g$ is continuous on $R$ and analytic in the interior of $R$ (by Theorem 2.18.1 and Lemma 2.24.B). Next,

$$|g(z)| = |e^{f(z)}| = |e^{u(x, y) + iv(x, y)}| = |e^{u(x, y)}| e^{iv(x, y)} = e^{u(x, y)}. \tag{1}$$

So by Corollary 4.54.D, $|g(z)| = e^{u(x, y)}$ attains the maximum on the boundary of $R$. Since $e^x$ is an increasing function of real variable $x$, then $u(x, y)$ attains its maximum at the same point on the boundary of $R$. Since $f$ is not a constant then the maximum $u(x, y)$ (and hence $|g(z)|$) cannot occur at an interior point also be Corollary 4.54.D.

\[ \square \]