Lemma 4.54.A

Suppose that $|f(z)| \leq |f(z_0)|$ at each point $z$ in some neighborhood $|z - z_0| < \varepsilon$ in which $f$ is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

**Proof.** Let $z_1 \neq z_0$ be in the $\varepsilon$-neighborhood of $z_0$. Let $\rho = |z_1 - z_0|$. Let $C_\rho$ be the positively oriented circle $|z - z_0| = \rho$. Then $f$ is analytic on and inside $C_\rho$ and so by the Cauchy Integral Formula (Theorem 4.50.A),

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z) \, dz}{z - z_0}.$$  

Parameterize $C_\rho$ as $z = z_0 + \rho e^{i\theta}$, $\theta \in [0, 2\pi]$. Then

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{(z_0 + \rho e^{i\theta}) - z_0} \rho e^{i\theta} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \, d\theta. \quad (2)$$

We then have from (2) that

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta$$

by Lemma 4.43.A. \hfill (3)
Theorem 4.54.C (continued)

Proof (continued).

Since \(|f(z_0)|\) is a maximum of \(|f(z)|\) on \(D\), then it is a maximum on \(N_0\) and \(z_0 \in N_0\), so by Lemma 4.54.A, \(f\) is constant on \(N_0\). In particular, \(f(z_1) = f(z_0)\). So \(|f(z_1)|\) is a maximum of \(|f(z)|\) on \(N_1\) and \(z_1 \in N_1\), so by Lemma 4.54.A, \(f\) is a constant on \(N_1\). Inductively, \(f\) is constant on \(N_0 \cup N_2 \cup \cdots \cup N_n\) and so \(f(z_0) = f(P)\). Since \(P\) is an arbitrary point of \(D\), then \(f\) is constant on \(D\), a CONTRADICTION. So the assumption that \(|f(z)|\) has a maximum on \(D\) is false, and \(|f(z)|\) has no maximum on \(D\), as claimed.

Theorem 4.54.D

**Theorem 4.54.D. Maximum Modulus Theorem, Alternative Version.** Suppose that a function \(f\) is continuous on a closed bounded region \(R\) and that it is analytic and not constant in the interior of \(R\). Then the maximum value of \(|f(z)|\) on \(R\), which is always reached (by Theorem 2.18.3) occurs somewhere on the boundary of \(R\) and never in the interior.

**Proof.** Let \(M\) be the maximum of \(|f(z)|\) on \(R\), so that \(|f(z)| \leq M\) for all \(z \in R\). If \(f\) is constant, then \(|f(z)| = M\) for all \(z \in R\) and so the maximum is attained on the boundary. If \(f\) is not constant, then by the Maximum Modulus Theorem (Theorem 4.54.C) the maximum of \(|f(z)|\) cannot be attained for some \(z_0\) in the interior of \(R\). Since the maximum is attained somewhere on \(R\) by Theorem 2.18.3, then it must be attained on the boundary of \(R\) (recall that a “region” is an open connected set along with some, none, or all of its boundary points).

Theorem 4.54.E

**Theorem 4.54.E.** Let \(f\) be continuous on a closed bounded region \(R\), and analytic and not constant on the interior of \(R\). For \(f(z) = u(x, y) + iv(x, y)\), where \(z = x + iy\), function \(u(x, y)\) attains its maximum value in \(R\) on the boundary of \(R\) and not in the interior.

**Proof.** Let \(g(z) = e^{f(z)}\). Then \(g\) is continuous on \(R\) and analytic in the interior of \(R\) (by Theorem 2.18.1 and Lemma 2.24.B). Next,

\[ |g(z)| = |e^{f(z)}| = |e^{u(x, y) + iv(x, y)}| = |e^{u(x, y)}||e^{iv(x, y)}| = e^{u(x, y)}. \]

So by Corollary 4.54.D, \(|g(z)| = e^{u(x, y)}\) attains the maximum on the boundary of \(R\). Since \(e^x\) is an increasing function of real variable \(x\), then \(u(x, y)\) attains its maximum at the same point on the boundary of \(R\). Since \(f\) is not a constant then the maximum \(u(x, y)\) (and hence \(|g(z)|\)) cannot occur at an interior point also by Corollary 4.54.D.