Chapter 4. Integrals
Section 4.54. Maximum Modulus Principle—Proofs of Theorems
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Lemma 4.54.A

Lemma 4.54.A. Suppose that $|f(z)| \leq |f(z_0)|$ at each point $z$ in some neighborhood $|z - z_0| < \varepsilon$ in which $f$ is analytic. Then $|f(z)|$ has the constant value $f(z_0)$ throughout that neighborhood.

Proof. Let $z_1 \neq z_0$ be in the $\varepsilon$-neighborhood of $z_0$. Let $\rho = |z_1 - z_0|$. Let $C_\rho$ be the positively oriented circle $|z - z_0| = \rho$. Then $f$ is analytic on and inside $C_\rho$ and so by the Cauchy Integral Formula (Theorem 4.50.A),

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z) \, dz}{z - z_0}.$$
Lemma 4.54.A

**Lemma 4.54.A.** Suppose that \(|f(z)| \leq |f(z_0)|\) at each point \(z\) in some neighborhood \(|z - z_0| < \varepsilon\) in which \(f\) is analytic. Then \(|f(z)|\) has the constant value \(f(z_0)\) throughout that neighborhood.

**Proof.** Let \(z_1 \neq z_0\) be in the \(\varepsilon\)-neighborhood of \(z_0\). Let \(\rho = |z_1 - z_0|\). Let \(C_{\rho}\) be the positively oriented circle \(|z - z_0| = \rho\). Then \(f\) is analytic on and inside \(C_{\rho}\) and so by the Cauchy Integral Formula (Theorem 4.50.A),

\[
f(z_0) = \frac{1}{2\pi i} \int_{C_{\rho}} \frac{f(z)}{z - z_0} \, dz.
\]

Parameterize \(C_{\rho}\) as \(z = z_0 + \rho e^{i\theta}, \theta \in [0, 2\pi]\). Then

\[
f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{(z_0 + \rho e^{i\theta}) - z_0} i\rho e^{i\theta} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \, d\theta. \tag{2}
\]

we then have from (2) that

\[
|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta \text{ by Lemma 4.43.A.} \tag{3}
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Lemma 4.54.A

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**Proof.** Let $z_1 \neq z_0$ be in the $\varepsilon$-neighborhood of $z_0$. Let $\rho = |z_1 - z_0|$. Let $C_\rho$ be the positively oriented circle $|z - z_0| = \rho$. Then $f$ is analytic on and inside $C_\rho$ and so by the Cauchy Integral Formula (Theorem 4.50.A),

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} dz.$$  

Parameterize $C_\rho$ as $z = z_0 + \rho e^{i\theta}$, $\theta \in [0, 2\pi]$. Then

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{(z_0 + \rho e^{i\theta}) - z_0} i\rho e^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta. \quad (2)$$

we then have from (2) that

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta \text{ by Lemma 4.43.A.} \quad (3)$$
Proof (continued). On the other hand, by hypothesis,

\[ |f(z_0 + \rho e^{i\theta})| \leq |f(z_0)| \text{ for } \theta \in [0, 2\pi] \] so that

\[ \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta \leq \int_0^{2\pi} |f(z_0)| \, d\theta = 2\pi |f(z_0)|, \] or

\[ |f(z_0)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta. \] (5)
Lemma 4.51.A (continued)

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or

\[ |f(z_0)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta. \quad (5) \]

Combining equations (3) and (5) gives

\[ |f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta, \]

or

\[ \int_0^{2\pi} (|f(z_0)| - |f(z_0 + \rho e^{i\theta})|) \, d\theta = 0. \]

Now the integrand is nonnegative by hypothesis and is a continuous function of \( \theta \) on \([0, 2\pi]\). If the integral of a continuous real-valued nonnegative function over some interval is 0 then the function must be identically 0 (yes, we could use a reference for this).
Lemma 4.51.A (continued)

Proof (continued). On the other hand, by hypothesis, 
\[ |f(z_0 + \rho e^{i\theta})| \leq |f(z_0)| \quad \text{for } \theta \in [0, 2\pi] \] so that 
\[ \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta \leq \int_0^{2\pi} |f(z_0)| \, d\theta = 2\pi |f(z_0)|, \] or
\[ |f(z_0)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta. \quad (5) \]

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Now the integrand is nonnegative by hypothesis and is a continuous function of \( \theta \) on \([0, 2\pi]\). If the integral of a continuous real-valued nonnegative function over some interval is 0 then the function must be identically 0 (yes, we could use a reference for this). So 
\[ |f(z_0)| = |f(z_0 + \rho e^{i\theta})| \quad \text{for all } \theta \in [0, 2\pi]. \] That is, 
\[ |f(z_0)| = |f(z)| \quad \text{for all } z \in C_\rho. \] In particular, 
\[ |f(z_0)| = |f(z_1)|. \] Since \( z_1 \) is an arbitrary point in the \( \varepsilon \)-neighborhood of \( z_0 \), then 
\[ |f(z_0)| = |f(z)| \quad \text{for all } z \text{ such that } |z - z_0| < \varepsilon. \] So by Example 2.25.4/Theorem 2.25.B, 
\[ f(z) = f(z_0) \quad \text{for all } z \text{ satisfying } |z - z_0| < \varepsilon, \] as claimed. \( \square \)
Lemma 4.51.A (continued)

Proof (continued). On the other hand, by hypothesis,
\[ |f(z_0 + \rho e^{i\theta})| \leq |f(z_0)| \text{ for } \theta \in [0, 2\pi] \] so that
\[ \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta \leq \int_0^{2\pi} |f(z_0)| \, d\theta = 2\pi |f(z_0)|, \]
or
\[ |f(z_0)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta. \quad (5) \]
Combining equations (3) and (5) gives
\[ |f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta, \]
or
\[ \int_0^{2\pi} (|f(z_0)| - |f(z_0 + \rho e^{i\theta})|) \, d\theta = 0. \] Now the integrand is nonnegative by hypothesis and is a continuous function of \( \theta \) on \([0, 2\pi]\). If the integral of a continuous real-valued nonnegative function over some interval is 0 then the function must be identically 0 (yes, we could use a reference for this). So
\[ |f(z_0)| = |f(z_0 + \rho e^{i\theta})| \text{ for all } \theta \in [0, 2\pi]. \] That is,
\[ |f(z_0)| = |f(z)| \text{ for all } z \in C_\rho. \] In particular, \( |f(z_0)| = |f(z_1)|. \) Since \( z_1 \) is an arbitrary point in the \( \varepsilon \)-neighborhood of \( z_0 \), then \( |f(z_0)| = |f(z)| \) for all \( z \) such that \( |z - z_0| < \varepsilon \). So by Example 2.25.4/Theorem 2.25.B, \( f(z) = f(z_0) \) for all \( z \) satisfying \( |z - z_0| < \varepsilon \), as claimed. \( \square \)
Theorem 4.54.C. The Maximum Modulus Theorem.

If a function $f$ is analytic and not constant in a given domain $D$, then $|f(z)|$ has no maximum value in $D$. That is, there is no point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all points $z \in D$.

Proof. Let $f$ be analytic and nonconstant on $D$. ASSUME $|f(z)|$ has a maximum on $D$ of $|f(z_0)|$ for some $z_0 \in D$. 
Corollary 4.54.C. The Maximum Modulus Theorem

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Proof. Let $f$ be analytic and nonconstant on $D$. ASSUME $|f(z)|$ has a maximum on $D$ of $|f(z_0)|$ for some $z_0 \in D$. Let $P$ be any point in $D$. Let $L$ be a polygonal line lying in $D$ and joining $z_0$ and $P$ (such a polygonal line exists since $D$ is open and connected by Theorem II.2.3 in my online notes for Complex Analysis 1 [MATH 5510] on II.2. Connectedness. If $D \neq \mathbb{C}$, then let $d$ be the shortest distance from the points on $L$ to the boundary of $D$ (such $d$ exists by Theorem II.5.17 in my online Complex Analysis notes on II.5. Continuity). If $D = \mathbb{C}$, let $d = 1$. 
Theorem 4.54.C. The Maximum Modulus Theorem.

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Theorem 4.54.C. The Maximum Modulus Theorem.

If a function $f$ is analytic and not constant in a given domain $D$, then $|f(z)|$ has no maximum value in $D$. That is, there is no point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all points $z \in D$.

Proof. Let $f$ be analytic and nonconstant on $D$. Assume $|f(z)|$ has a maximum on $D$ of $|f(z_0)|$ for some $z_0 \in D$. Let $P$ be any point in $D$. Let $L$ be a polygonal line lying in $D$ and joining $z_0$ and $P$ (such a polygonal line exists since $D$ is open and connected by Theorem II.2.3 in my online notes for Complex Analysis 1 [MATH 5510] on II.2. Connectedness. If $D \neq \mathbb{C}$, then let $d$ be the shortest distance from the points on $L$ to the boundary of $D$ (such $d$ exists by Theorem II.5.17 in my online Complex Analysis notes on II.5. Continuity). If $D = \mathbb{C}$, let $d = 1$. Next, since $L$ is of finite length, there are complex numbers on $L$, $z_0, z_1, \ldots, z_n$ such that $|z_k - z_{k-1}| < d$ for $k = 1, 2, \ldots, n$. Define neighborhood $N_k$ of $z_k$ as $N_k = \{z \in \mathbb{C} | |z - z_k| < d\}$. Then the $N_k$ are all subsets of domain $D$, and the center of $N_k$ lies in $N_{k-1}$ for $k = 1, 2, \ldots, n$. See Figure 71.
Theorem 4.54.C (continued)

Proof (continued).

Since $|f(z_0)|$ is a maximum of $|f(z)|$ on $D$, then it is a maximum on $N_0$ and $z_0 \in N_0$, so by Lemma 4.54.A, $f$ is constant on $N_0$. In particular, $f(z_1) = f(z_0)$. So $|f(z_1)|$ is a maximum of $|f(z)|$ on $N_1$ and $z_1 \in N_1$, so by Lemma 4.54.A, $f$ is a constant on $N_1$. Inductively, $f$ is constant on $N_0 \cup N_1 \cup \cdots \cup N_n$ and so $f(z_0) = f(P)$. Since $P$ is an arbitrary point of $D$, then $f$ is constant on $D$, as claimed.
Proof (continued).

Since $|f(z_0)|$ is a maximum of $|f(z)|$ on $D$, then it is a maximum on $N_0$ and $z_0 \in N_0$, so by Lemma 4.54.A, $f$ is constant on $N_0$. In particular, $f(z_1) = f(z_0)$. So $|f(z_1)|$ is a maximum of $|f(z)|$ on $N_1$ and $z_1 \in N_1$, so by Lemma 4.54.A, $f$ is a constant on $N_1$. Inductively, $f$ is constant on $N_0 \cup N_1 \cup \cdots \cup N_n$ and so $f(z_0) = f(P)$. Since $P$ is an arbitrary point of $D$, then $f$ is constant on $D$, as claimed.

Suppose that a function $f$ is continuous on a closed bounded region $R$ and that it is analytic and not constant in the interior of $R$. Then the maximum value of $|f(z)|$ on $R$, which is always reached (by Theorem 2.18.3) occurs somewhere on the boundary of $R$ and never in the interior.

Proof. Let $M$ be the maximum of $|f(z)|$ on $R$, so that $|f(z)| \leq M$ for all $z \in R$. If $f$ is constant, then $|f(z)| = M$ for all $z \in R$ and so the maximum is attained on the boundary.
Suppose that a function \( f \) is continuous on a closed bounded region \( R \) and that it is analytic and not constant in the interior of \( R \). Then the maximum value of \( |f(z)| \) on \( R \), which is always reached (by Theorem 2.18.3) occurs somewhere on the boundary of \( R \) and never in the interior.

Proof. Let \( M \) be the maximum of \( |f(z)| \) on \( R \), so that \( |f(z)| \leq M \) for all \( z \in R \). If \( f \) is constant, then \( |f(z)| = M \) for all \( z \in R \) and so the maximum is attained on the boundary. If \( f \) is not constant, then by the Maximum Modulus Theorem (Theorem 4.54.C) the maximum of \( |f(z)| \) cannot be attained for some \( z_0 \) in the interior of \( R \). Since the maximum is attained somewhere on \( R \) by Theorem 2.18.3, then it must be attained on the boundary of \( R \) (recall that a “region” is an open connected set along with some, none, or all of its boundary points).

Suppose that a function $f$ is continuous on a closed bounded region $R$ and that it is analytic and not constant in the interior of $R$. Then the maximum value of $|f(z)|$ on $R$, which is always reached (by Theorem 2.18.3) occurs somewhere on the boundary of $R$ and never in the interior.

Proof. Let $M$ be the maximum of $|f(z)|$ on $R$, so that $|f(z)| \leq M$ for all $z \in R$. If $f$ is constant, then $|f(z)| = M$ for all $z \in R$ and so the maximum is attained on the boundary. If $f$ is not constant, then by the Maximum Modulus Theorem (Theorem 4.54.C) the maximum of $|f(z)|$ cannot be attained for some $z_0$ in the interior of $R$. Since the maximum is attained somewhere on $R$ by Theorem 2.18.3, then it must be attained on the boundary of $R$ (recall that a “region” is an open connected set along with some, none, or all of its boundary points).
Theorem 4.54.E. Let $f$ be continuous on a closed bounded region $R$, and analytic and not constant on the interior of $R$. For $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, function $u(x, y)$ attains its maximum value in $R$ on the boundary of $R$ and not in the interior.

Proof. Let $g(z) = e^{f(z)}$. Then $g$ is continuous on $R$ and analytic in the interior of $R$ (by Theorem 2.18.1 and Lemma 2.24.B). Next,

$$|g(z)| = |e^{f(z)}| = |f u(x, y) + iv(x, y)| = |e^{u(x, y)}||e^{iv(x, y)}| = e^{u(x, y)}.$$
Theorem 4.54.E

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So by Corollary 4.54.D, $|g(z)| = e^{u(x,y)}$ attains the maximum on the boundary of $R$. Since $e^x$ is an increasing function of real variable $x$, then $u(x, y)$ attains its maximum at the same point on the boundary of $R$. Since $f$ is not a constant then the maximum $u(x, y)$ (and hence $|g(z)|$) cannot occur at an interior point also be Corollary 4.54.D.
Theorem 4.54.E

**Theorem 4.54.E.** Let $f$ be continuous on a closed bounded region $R$, and analytic and not constant on the interior of $R$. For $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, function $u(x, y)$ attains its maximum value in $R$ on the boundary of $R$ and not in the interior.

**Proof.** Let $g(z) = e^{f(z)}$. Then $g$ is continuous on $R$ and analytic in the interior of $R$ (by Theorem 2.18.1 and Lemma 2.24.B). Next,

$$|g(z)| = |e^{f(z)}| = |f u(x,y) + iv(x,y)| = |e^{u(x,y)}| |e^{iv(x,y)}| = e^{u(x,y)}.$$

So by Corollary 4.54.D, $|g(z)| = e^{u(x,y)}$ attains the maximum on the boundary of $R$. Since $e^x$ is an increasing function of real variable $x$, then $u(x,y)$ attains its maximum at the same point on the boundary of $R$. Since $f$ is not a constant then the maximum $u(x,y)$ (and hence $|g(z)|$) cannot occur at an interior point also be Corollary 4.54.D. \qed