Chapter 5. Series
Section 5.56. Convergence of Series—Proofs of Theorems

Theorem 5.56.A

Theorem 5.56.A. Suppose that \( z_n = x_n + iy_n \) and \( S = X + iY \). Then \( \sum_{n=1}^{\infty} z_n = S \) if and only if \( \sum_{n=1}^{\infty} x_n = X \) and \( \sum_{n=1}^{\infty} y_n = Y \).

Proof. Let \( X_N = \sum_{n=1}^{N} x_n \) and \( Y_N = \sum_{n=1}^{N} y_n \). Then

\[
S_N = \sum_{n=1}^{N} z_n = \sum_{n=1}^{N} (x_n + iy_n) = \sum_{n=1}^{N} x_n + i\sum_{n=1}^{N} y_n = X_N + iY_N.
\]

So \( \sum_{n=1}^{\infty} z_n = S \) if and only if \( \lim_{n \to \infty} S_N = S \); that is, if and only if

\[
\lim_{n \to \infty} (X_n + iY_n) = S. \]

Now by Theorem 5.55.A,

\[
\lim_{n \to \infty} (X_n + iY_n) = \lim_{n \to \infty} X_n + i\lim_{n \to \infty} Y_n = X + iY.
\]

So \( \sum_{n=1}^{\infty} z_n = S \) if and only if \( S = X + iY \), as claimed. \( \Box \)

Corollary 5.56.1. Test for Divergence

If a series of complex numbers converges, then the \( n \)th term converges to zero as \( n \) tends to infinity. That is, if \( z_n \) does not converge to 0 then \( \sum_{n=1}^{\infty} z_n \) diverges.

Proof. Let \( \sum_{n=1}^{\infty} z_n \) converge. With \( z_n = x_n + iy_n \), Theorem 5.56.A implies that \( \sum_{n=1}^{\infty} x_n \) and \( \sum_{n=1}^{\infty} y_n \) both converge. By the Test for Divergence from Calculus 2 (see Theorem 7 of my online notes on 10.2. Infinite Series), we have that \( x_n \) converges to 0 and \( y_n \) converges to 0. So by Theorem 5.55.A,

\[
\lim_{n \to \infty} z_n = \lim_{n \to \infty} x_n + i\lim_{n \to \infty} y_n = 0 + 0 = 0.
\]

Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof. Suppose series \( \sum_{n=0}^{\infty} z_n \) converges absolutely. With \( z_n = x_n + iy_n \), we have

\[
|x_n| = \sqrt{x_n^2 + y_n^2} = |z_n| \quad \text{and} \quad |y_n| = \sqrt{y_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|.
\]

So by the Direct Comparison Test for series of real numbers from Calculus 2 (see Theorem 10. The (Direct) Comparison Test in my online notes on 10.4. Comparison Tests), we see that both \( \sum_{n=0}^{\infty} |x_n| \) and \( \sum_{n=0}^{\infty} |y_n| \) converge.

So both the series of real numbers \( \sum_{n=0}^{\infty} x_n \) and \( \sum_{n=0}^{\infty} y_n \) converge absolutely.
Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof (continued). Since the absolute convergence of a series of real numbers implies its convergence (see Theorem 16. The Absolute Convergence Theorem in my online Calculus 2 notes on 10.6. Alternating Series, Absolute and Conditional Convergence), then the series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ both converge. Therefore, by Theorem 5.56.A, the series $\sum_{n=0}^{\infty} z_n$ converges. \qed