Chapter 5. Series
Section 5.56. Convergence of Series—Proofs of Theorems
1 Theorem 5.56.A

2 Corollary 5.56.1. Test for Divergence

3 Corollary 5.56.2
Theorem 5.56.A. Suppose that $z_n = x_n + iy_n$ and $S = X + iY$. Then $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

Proof. Let $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$. Then

$$S_N = \sum_{n=1}^{N} z_n = \sum_{n=1}^{N} (x_n + iy_n) = \sum_{n=1}^{N} x_n + i \sum_{n=1}^{N} y_n = X_N + iY_N.$$
Theorem 5.56.A. Suppose that \( z_n = x_n + iy_n \) and \( S = X + iY \). Then \( \sum_{n=1}^{\infty} z_n = S \) if and only if \( \sum_{n=1}^{\infty} x_n = X \) and \( \sum_{n=1}^{\infty} y_n = Y \).

Proof. Let \( X_N = \sum_{n=1}^{N} x_n \) and \( Y_N = \sum_{n=1}^{N} y_n \). Then

\[
S_N = \sum_{n=1}^{N} z_n = \sum_{n=1}^{N} (x_n + iy_n) = \sum_{n=1}^{N} x_n + i\sum_{n=1}^{N} y_n = X_N + iY_N.
\]

So \( \sum_{n=1}^{\infty} z_n = S \) if and only if \( \lim_{n \to \infty} S_N = S \); that is, if and only if \( \lim_{n \to \infty} (X_n + iY_n) = S \). Now by Theorem 5.55.A,

\[
\lim_{n \to \infty} (X_n + iY_n) = \lim_{n \to \infty} X_n + i \lim_{n \to \infty} Y_n = X + iY.
\]

So \( \sum_{n=1}^{\infty} z_n = S \) if and only if \( S = X + iY \), as claimed.
Theorem 5.56.A. Suppose that $z_n = x_n + iy_n$ and $S = X + iY$. Then \( \sum_{n=1}^{\infty} z_n = S \) if and only if \( \sum_{n=1}^{\infty} x_n = X \) and \( \sum_{n=1}^{\infty} y_n = Y \).

Proof. Let $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$. Then

\[
S_N = \sum_{n=1}^{N} z_n = \sum_{n=1}^{N} (x_n + iy_n) = \sum_{n=1}^{N} x_n + i \sum_{n=1}^{N} y_n = X_N + iY_N.
\]

So $\sum_{n=1}^{\infty} z_n = S$ if and only if $\lim_{n \to \infty} S_N = S$; that is, if and only if $\lim_{n \to \infty} (X_n + iY_n) = S$. Now by Theorem 5.55.A, $\lim_{n \to \infty} (X_n + iY_n) = \lim_{n \to \infty} X_n + i \lim_{n \to \infty} Y_n = X + iY$. So $\sum_{n=1}^{\infty} z_n = S$ if and only if $S = X + iY$, as claimed. \(\square\)
Corollary 5.56.1. Test for Divergence

If a series of complex numbers converges, then the \( n \)th term converges to zero as \( n \) tends to infinity. That is, if \( z_n \) does not converge to 0 then \( \sum_{n=1}^{\infty} z_n \) diverges.

Proof. Let \( \sum_{n=1}^{\infty} z_n \) converge. With \( z_n = x_n + iy_n \), Theorem 5.56.A implies that \( \sum_{n=1}^{\infty} x_n \) and \( \sum_{n=1}^{\infty} y_n \) both converge.
Corollary 5.56.1. Test for Divergence
If a series of complex numbers converges, then the $n$th term converges to zero as $n$ tends to infinity. That is, if $z_n$ does not converge to 0 then $\sum_{n=1}^{\infty} z_n$ diverges.

Proof. Let $\sum_{n=1}^{\infty} z_n$ converge. With $z_n = x_n + iy_n$, Theorem 5.56.A implies that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ both converge. By the Test for Divergence from Calculus 2 (see Theorem 7 of my online notes on 10.2. Infinite Series), we have that $x_n$ converges to 0 and $y_n$ converges to 0. So by Theorem 5.55.A,

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} x_n + i \lim_{n \to \infty} y_n = 0 + i0 = 0.$$
Corollary 5.56.1. Test for Divergence

If a series of complex numbers converges, then the \( n \)th term converges to zero as \( n \) tends to infinity. That is, if \( z_n \) does not converge to 0 then \( \sum_{n=1}^{\infty} z_n \) diverges.

**Proof.** Let \( \sum_{n=1}^{\infty} z_n \) converge. With \( z_n = x_n + iy_n \), Theorem 5.56.A implies that \( \sum_{n=1}^{\infty} x_n \) and \( \sum_{n=1}^{\infty} y_n \) both converge. By the Test for Divergence from Calculus 2 (see Theorem 7 of my online notes on 10.2. Infinite Series), we have that \( x_n \) converges to 0 and \( y_n \) converges to 0. So by Theorem 5.55.A,

\[
\lim_{n \to \infty} z_n = \lim_{n \to \infty} x_n + i \lim_{n \to \infty} y_n = 0 + i0 = 0.
\]
Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof. Suppose series $\sum_{n=0}^{\infty} z_n$ converges absolutely. With $z_n = x_n + iy_n$, we have $|x_n| = \sqrt{x_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$ and $|y_n| = \sqrt{y_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$.
Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof. Suppose series $\sum_{n=0}^{\infty} z_n$ converges absolutely. With $z_n = x_n + iy_n$, we have $|x_n| = \sqrt{x_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$ and $|y_n| = \sqrt{y_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$. So by the Direct Comparison Test for series of real numbers from Calculus 2 (see Theorem 10. The (Direct) Comparison Test in my online notes on 10.4. Comparison Tests), we see that both $\sum_{n=0}^{\infty} |x_n|$ and $\sum_{n=0}^{\infty} |y_n|$ converge. So both the series of real numbers $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ converge absolutely.
Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof. Suppose series $\sum_{n=0}^{\infty} z_n$ converges absolutely. With $z_n = x_n + iy_n$, we have $|x_n| = \sqrt{x_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$ and $|y_n| = \sqrt{y_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$. So by the Direct Comparison Test for series of real numbers from Calculus 2 (see Theorem 10. The (Direct) Comparison Test in my online notes on 10.4. Comparison Tests), we see that both $\sum_{n=0}^{\infty} |x_n|$ and $\sum_{n=0}^{\infty} |y_n|$ converge. So both the series of real numbers $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ converge absolutely.
Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof (continued. Since the absolute convergence of a series of real numbers implies its convergence (see Theorem 16. The Absolute Convergence Theorem in my online Calculus 2 notes on 10.6. Alternating Series, Absolute and Conditional Convergence), then the series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ both converge.
Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof (continued). Since the absolute convergence of a series of real numbers implies its convergence (see Theorem 16. The Absolute Convergence Theorem in my online Calculus 2 notes on 10.6. Alternating Series, Absolute and Conditional Convergence), then the series \( \sum_{n=0}^{\infty} x_n \) and \( \sum_{n=0}^{\infty} y_n \) both converge. Therefore, by Theorem 5.56.A, the series \( \sum_{n=0}^{\infty} z_n \) converges.
Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof (continued). Since the absolute convergence of a series of real numbers implies its convergence (see Theorem 16. The Absolute Convergence Theorem in my online Calculus 2 notes on 10.6. Alternating Series, Absolute and Conditional Convergence), then the series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ both converge. Therefore, by Theorem 5.56.A, the series $\sum_{n=0}^{\infty} z_n$ converges.