Theorem 5.64.1

Theorem 5.64.1. A power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z - z_0| = R$. 

Proof. Let $S_N(z) = \sum_{n=0}^{N-1} a_n(z - z_0)^n$ and consider the remainder function $\rho_N(z) = S(z) - S_N(z)$ for $|z - z_0| < R$. Then, because $S(z) = S_N(z) + \rho_N(z)$ for $|z - z_0| < R$, we have

$$|S(z) - S(z_1)| = |(S_N(z) + \rho_N(z)) - (S_N(z_1) + \rho_N(z_1))|$$

$$\leq |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)| \text{ by the Triangle Inequality. (}*$$

Let $\epsilon > 0$. If $z$ is any point in some closed disk $|z - z_0| \leq R_0$ where $|z_1 - z_0| < R_0 < R_1$, then by the uniform convergence of the power series on set $|z - z_0| \leq R_0$ as given by Theorem 5.63.2, there is $N_\epsilon \in \mathbb{N}$ such that

$$|\rho_N(z)| < \frac{\epsilon}{3} \text{ whenever } N > N_\epsilon. \text{ (**)$$

Proof (continued). In particular, this inequality holds for each point $z$ in some neighborhood $|z - z_1| < \delta_1$ of $z_1$ that is small enough to be contained in the disk $|z - z_1| \leq R_0$ (see Figure 81).

![Figure 81](image)

Now the partial sum $S_N(z)$ is a polynomial and so is continuous for each value of $N$ at $z = z_1$ by Corollary 2.18.B. When $N = N_\epsilon + 1$, by the definitions of continuity and limit, we can choose $\delta_2 > 0$ such that

$$|S_N(z) - S_N(z_1)| < \frac{\epsilon}{3} \text{ whenever } |z - z_1| < \delta_2. \text{ (**)$$

Therefore $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is continuous at $z_1$ and, since $z_1$ is an arbitrary point inside the circle of convergence, $S(z)$ is continuous inside the circle of convergence, as claimed. □