Chapter 5. Series
Section 5.65. Integration and Differentiation of Power Series—Proofs of Theorems
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Theorem 5.65.1

**Theorem 5.65.1.** Let $C$ denote any contour interior to the circle of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and let $g(z)$ be any function that is continuous on $C$. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term-by-term over $C$; that is

$$\int_C g(z)S(z)\,dz = \int_C g(z)\sum_{n=0}^{\infty} a_n(z - z_0)^n\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\,dz.$$

**Proof.** Notice that $g(z)$ is continuous on $C$ by hypothesis and $S(z)$ is continuous on $C$ by Theorem 5.64.1, so $\int_C g(z)S(z)\,dz$ is defined. With $\rho_N(z)$ as the remainder $S(z) - S_N(z)$ (where $S_N(z)$ is the $N$th partial sum), by Note 4.40.A, ...
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Theorem 5.65.1 (continued 1)

Proof (continued).

\[\int_C g(z)S(z) \, dz = \int_C g(z)(S_N(z) + \rho_N(z)) \, dz\]

\[= \int_C \left( g(z) \sum_{n=0}^{N-1} a_n(z - z_0)^n + g(z)\rho_N(z) \right) \, dz\]

\[= \sum_{n=1}^{N-1} a_n \int_C g(z)(z - z_0)^n \, dz + \int_C g(z)\rho_N(z) \, dz. \quad (\ast)\]

Now \(|g(z)|\) has a maximum \(M\) on \(C\) by Theorem 2.18.3. Let \(L\) denote the length of \(C\). Since power series \(\rho_N(z)\) is uniformly convergent on \(C\) by Theorem 5.64.2 so for any \(\varepsilon > 0\) there exists \(N_\varepsilon \in \mathbb{N}\) such that \(|\rho_N(z)| < \varepsilon\) for all \(N > N_\varepsilon\) and for all \(z \in C\).
Theorem 5.65.1 (continued 1)

Proof (continued).

\[ \int_C g(z) S(z) \, dz = \int_C g(z) (S_N(z) + \rho_N(z)) \, dz \]
\[ = \int_C \left( g(z) \sum_{n=0}^{N-1} a_n (z - z_0)^n + g(z) \rho_N(z) \right) \, dz \]
\[ = \sum_{n=1}^{N-1} a_n \int_C g(z)(z - z_0)^n \, dz + \int_C g(z)\rho_N(z) \, dz. \quad (*) \]

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\(|\rho_N(z)| < \varepsilon\) for all \(N > N_{\varepsilon}\) and for all \(z \in C\).
Theorem 5.65.1 (continued 2)

**Proof (continued).** So (since $N_\varepsilon$ is independent of $z \in C$) we have

$$\left| \int_C g(z) \rho_N(z) \, dz \right| < M\varepsilon L \text{ whenever } N > N_\varepsilon$$

by Theorem 4.43.A. Since $M$ and $L$ are constant (because $C$ is given) then this last condition implies that $\lim_{N \to \infty} \int_C g(z) \rho_N(z) \, dz = 0$ by the definition of limit. Taking a limit as $N \to \infty$ for both sides of $(\star)$ we get

$$\lim_{N \to \infty} \left( \int_C g(z) S(z) \, dz \right)$$

$$= \lim_{N \to \infty} \left( \sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n \, dz + \int_C g(z) \rho_N(z) \, dz \right)$$

or . . .
Theorem 5.65.1 (continued 2)

**Proof (continued).** So (since $N_\varepsilon$ is independent of $z \in C$) we have

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Proof (continued). . . .

$$\int_C g(z)S(z) \, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n \, dz,$$

as claimed.
Corollary 5.65.1

**Corollary 5.65.1.** The power series \( S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) is analytic at each point \( z \) interior to the circle of convergence of the series.

**Proof.** Let \( C \) be any closed contour in the domain which is the interior of the circle of convergence. With \( g(z) = 1 \) we then have

\[
\int_C g(z)(z - z_0)^n \, dz = \int_C (z - z_0)^n \, dz = 0 \text{ for } n \in \mathbb{N} \cup \{0\}
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by Theorem 4.44.A (or Example 4.43.A).
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by Theorem 4.44.A (or Example 4.43.A). So by Theorem 5.65.1,

\[
\int_C g(z)S(z) \, dz = \int_C \sum_{n=0}^{\infty} a_n(z - z_0)^n \, dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n \, dz = 0.
\]

Since \( C \) is an arbitrary closed contour in the circle of convergence of the series, then by Morera’s Theorem (Theorem 4.52.2), \( S(z) = \sum_{n=0}^{\infty}(z - z_0)^n \) is analytic in the circle of convergence, as claimed.
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$$\int_C g(z)S(z) \, dz = \int_C \sum_{n=0}^{\infty} a_n(z - z_0)^n \, dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n \, dz = 0.$$

Since $C$ is an arbitrary closed contour in the circle of convergence of the series, then by Morera’s Theorem (Theorem 4.52.2), $S(z) = \sum_{n=0}^{\infty}(z - z_0)^n$ is analytic in the circle of convergence, as claimed.
Theorem 5.65.2. The power series $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ can be differentiated term-by-term in its circle of convergence. That is, at each point $z$ interior to the circle of convergence of that series, we have

$$S'(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}.$$ 

**Proof.** Let $z$ be any point interior to the circle of convergence of the series, and let $C$ be some positively oriented simple closed contour surrounding $z$ and interior to the circle. Define $g(s) = \frac{1}{2\pi i} \frac{1}{(s - z)^2}$ for each $s \in C$. Since $z \notin C$ then $g$ is continuous on $C$ (as is $S(z)$), so by Theorem 5.65.1

$$\int_C g(s)S(s) \, ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s - z_0)^n \, ds. \quad (*)$$
Theorem 5.65.2.

The power series \( S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) can be differentiated term-by-term in its circle of convergence. That is, at each point \( z \) interior to the circle of convergence of that series, we have

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\[
\int_{C} g(s)S(s) \, ds = \sum_{n=0}^{\infty} a_n \int_{C} g(s)(s - z_0)^n \, ds. \tag{\ast}
\]
Theorem 5.65.2 (continued)

Proof (continued). Since \( S(z) \) is analytic inside and on \( C \) by Corollary 5.65.1, then by Theorem 4.51.1 with \( n = 1 \) (and \( z_0 \) in the theorem as \( z \) here) we have

\[
S'(z) = \frac{1}{2\pi i} \int_C \frac{S(s) \, ds}{(s - z)^2} = \int_C g(s)S(s) \, ds. \tag{**}
\]

Similarly, replacing \( S(z) \) with \( (z - z_0)^n \) in (**) we have

\[
\frac{d}{dz}[(z - z_0)^n] = \frac{1}{2\pi i} \int_C \frac{(s - z_0)^n \, ds}{(s - z)^2} = \int_C g(s)(s - z_0)^n \, ds,
\]

and so, combining (*) and (**), we have

\[
\frac{d}{dz} \left[ \sum_{n=0}^{\infty} a_n(z - z_0)^n \right] = S'(z) = \int C g(s)S(s) \, ds = \sum_{n=0}^{\infty} a_n \int C g(s)(s - z_0)^n \, ds
\]

\[
= \sum_{n=0}^{\infty} a_n \frac{d}{dz}[(z - z_0)^n] = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}, \text{ as claimed.} \]
Theorem 5.65.2 (continued)

**Proof (continued).** Since $S(z)$ is analytic inside and on $C$ by Corollary 5.65.1, then by Theorem 4.51.1 with $n = 1$ (and $z_0$ in the theorem as $z$ here) we have

$$S'(z) = \frac{1}{2\pi i} \int_C \frac{S(s) \, ds}{(s - z)^2} = \int_C g(s) S(s) \, ds. \quad (***)$$

Similarly, replacing $S(z)$ with $(z - z_0)^n$ in (***), we have

$$\frac{d}{dz} [(z - z_0)^n] = \frac{1}{2\pi i} \int_C \frac{(s - z_0)^n \, ds}{(s - z)^2} = \int_C g(s)(s - z_0)^n \, ds,$$

and so, combining (*) and (**), we have

$$\frac{d}{dz} \left[ \sum_{n=0}^{\infty} a_n (z - z_0)^n \right] = S'(z) = \int_C g(s) S(s) \, ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s - z_0)^n \, ds$$

$$= \sum_{n=0}^{\infty} a_n \frac{d}{dz} [(z - z_0)^n] = \sum_{n=1}^{\infty} na_n (z - z_0)^{n-1}, \text{ as claimed}. \quad \square$$